Zurich University
of Applied Sciences

School of
Engineering
IEFE Institute of Energy Systems
and Fluid Engineering

IEFE Energy Papers, 11 (2022), Nr. 3

Main Equations and Thermal Characteristics of Radiators

Date: 3rd May 2022

Authors: Zhengji Zhang

Ciril Schiess

Frank Tillenkamp

Contact: zhengji.zhang@zhaw.ch

Legal Notice

Zurich University of Applied Sciences
School of Engineering
IEFE Institute of Energy Systems and Fluid Engineering
Technikumstrasse 9
CH-8401 Winterthur

www.zhaw.ch/iefe
info.iefe@zhaw.ch

DOI: 10.21256/zhaw-2536

Main Equations and Thermal Characteristics of Radiators

Zh. Zhang*, C. Schiess and F. Tillenkamp

ZHAW Zurich University of Applied Sciences, Institute of Energy Systems and Fluid Engineering (IEFE)
ciril.schiess@zhaw.ch; frank.tillenkamp@zhaw.ch

* Correspondence: zhengji.zhang@zhaw.ch

Abstract Accurate calculations of radiator thermal performance were performed based on accurate considerations of the varying heat transfer coefficient on the radiator surface. Using a reference point, both the main equation for heat power output and the main equation for exit temperature were derived. Both equations allow the prediction of all other operating points, so that the complete thermal characteristics of a given radiator can be calculated directly from the reference point without having to consider the geometrical parameters of the radiator and the physical properties of the air under free convection. For performing the load regulation with the exit flow temperature as the target, e.g., reverse calculations for the exact setting of the mass flow were presented. Furthermore, the approaching equation contained in DIN EN442 was reviewed for its accuracy and application limitations. For designs and calculations, both the eigen-constant and the eigen-parameter were furthermore introduced, which greatly facilitates the experimental determination of the heat transfer coefficient. The established relationship between the power output and the surface area of radiators showed satisfactory agreement with the data available in the VDI Heat Atlas.

Key words: Radiator; heat exchanger; main equation; DIN EN442; thermal characteristics; eigen-constant; eigen-parameter; series connection of radiators

1 Introduction

Among many types of heat exchangers, radiators are mainly used in heating, ventilation, and air conditioning (HVAC) systems, as well as in cooling systems of automobiles and electronics. Radiator operation is often subject to load control, often through mass flow regulation. In process controls in thermal engineering, load regulation of radiators can also be accomplished by changing the inlet temperature of the working fluid. The basis for load regulation of radiators is the thermal characteristic of the radiator, which is represented by the heat power output ($\dot{Q}$) as a function of the mass flow rate ($\dot{m}$) and the inlet temperature (T') of the working medium at a given ambient air temperature:

$$\dot{Q} = \mathrm{f}\left(\dot{m}, T', \text{geometry,fluid property}\right). \qquad (1)$$

This equation would be more applicable to radiators in a control system if the mass flow rate and the heating power can be related to the respective known reference values $\dot{m}_\mathrm{R}$ and $\dot{Q}_\mathrm{R}$, for instance, in the following form

$$\frac{\dot{Q}}{\dot{Q}_\mathrm{R}} = \mathrm{f}\left(\frac{\dot{m}}{\dot{m}_\mathrm{R}}, T'\right). \qquad (2)$$

Such a dependence has been lacking, despite the long history of applications and long-term developments. Both the design and the optimization of radiators are based solely on experimental data or empirical relationships. For radiators of different designs and configurations, corresponding significant data for thermal performances are commonly given by the manufactures, although often only for a reference point. For basic design and calculations, corresponding fundamentals can be found, for instance, in [**1, 2**].

The thermal characteristics of radiators in the form of Eq. (1) are based entirely on calculations of fundamental free convections. As shown in [**3**], not only the geometrical parameters but also the fluid properties must be taken into account each time. On the one hand, this represents a rather cumbersome treatment. On the other hand, the calculations suffer from computational inaccuracy due to the purely theoretical nature of calculations and thus the neglect of some real factors influencing the heat transfers. In contrast, the thermal characteristic in the form of Eq. (2) always refers to a known reference operating point which is obtained, for instance, from measurements. From this point of view, Eq. (2) would be much more applicable than Eq. (1) if it were available. This will be explained in detail below in section 2.

To estimate the power output of radiators, an indirect approaching method is proposed in the European standard DIN EN442. Instead of the mass flow rate in Eq. (2), the ratio of the mean temperature difference to the value under the nominal condition is considered as an independent variable. The related computational inaccuracy, however, could not yet be evaluated and thus remains unknown.

For the above-mentioned reasons, the present paper attempts to reveal the associated relations and to find solutions. It therefore consists of two parts. The first part is restricted to the establishment of the main equation of the radiators, which allows the direct calculation of all other operating points of a given radiator by knowing only one operating point (reference point). In other words, Eq. (2) can be accurately derived. The second part reveals the inaccuracy of the estimation equation proposed in DIN EN442 for radiators in HVAC applications. The inaccuracy also exists when using the logarithmic temperature difference.

It should be mentioned that radiators themselves, especially in HVAC applications, have been considered as a kind of mature apparatus and therefore there were almost no extended studies on thermal characteristics and related calculations for them. The approach in DIN EN442 was simply accepted as a standard equation and is often used by all radiator manufactures for providing data sheets.

2 Main Equations and Thermal Characteristics of Radiators

2.1 Fundamentals

In the present paper, the notations of almost all thermal parameters are in accordance with the notations found in the VDI Heat Atlas [**4**].

Despite the name of the apparatus, the dominant heat transfer mechanism in most radiators is convection rather than thermal radiation. In HVAC applications of radiators, for instance, heat exchange is based on both convective heat transfer on the fluid flow side and free convection to the ambient air. Because of the high heat transfer coefficient, i.e., the negligible thermal resistance in the radiator, the temperature on the outer surface of the radiator is comparable to the temperature of the working medium. The radiator itself thus behaves like a heat source for its environment, in which, as in buildings, free convective heat transfer dominates compared to thermal radiation.

The convective heat transfer between the radiator and the air in the environment is basically calculated by $\dot{Q} = \alpha A \Delta T$. The heat transfer coefficient itself depends on the thermal conductivity of the surface film and thus on the temperature difference between the radiator and the ambient air. This implies that the heat transfer coefficient is a variable along the radiator surface in the flow direction. In addition, the heat transfer coefficient at radiators in buildings also depends on whether the surface for free convection is horizontal or vertical. According to literature (e.g. [**5**]), the heat transfer coefficient can be shown proportional to $\mathrm{Ra}^{1/3}$ or $\mathrm{Ra}^{1/4}$, with Ra as the Rayleigh number. In term of the Nusselt number, it is expressed for instance as follows:

$$\mathrm{Nu} = \frac{\alpha l}{\lambda} = C \mathrm{Ra}^{1/3} . \tag{3}$$

The Rayleigh number is defined, in proportion to the temperature difference ΔT , as

$$\mathrm{Ra} = \frac{g\beta}{\nu^2} \Delta T l^3 \,\mathrm{Pr} . \tag{4}$$

Here, β, ν and λ represent respectively the thermal expansion coefficient, the kinematic viscosity and the thermal conductivity of the air in the environment. Together with the Prandtl number Pr, they can all be considered constant when calculating the thermal characteristics in practical applications. The characteristic length is denoted by *l*.

The convective heat transfer can then be expressed as $\dot{Q} = \alpha A \Delta T \sim \Delta T^n$ with $n = 1 + 1/3$ or $n = 1 + 1/4$, depending on the Rayleigh number. It should be noted that the temperature difference ΔT and thus both the heat transfer coefficient α and the heat flux $\mathrm{d}\dot{Q}/\mathrm{d}A$ basically change along the radiator surface in the direction of flow. For the overall heat power output of a radiator, integration calculations must be carried out.

Since the heat transfer coefficient α on the radiator is not constant, the calculation of its mean value should in principle and due to $\dot{Q} = \alpha A \Delta T$ always be connected with the calculation of an appropriate temperature difference. This means that for each mean temperature difference (logarithmic or arithmetic) there exists a mean heat transfer coefficient α. As shown in [3] using examples with three different mean temperature differences, this obviously represents a highly inconvenient way to calculate the overall power output of a given radiator.

Therefore, the accurate calculation of radiators relies on the combined consideration of the heat transfer coefficient and the temperature difference. In other words, the two parameters should not be treated separately.

As a starting point for calculations, the energy balance along the radiator in the direction of flow can be written as

$$\mathrm{d}\dot{Q} = -c_\mathrm{p}\dot{m}\mathrm{d}T = \alpha\left(T - T_\mathrm{air}\right)\mathrm{d}A\,. \tag{5}$$

The heat transfer coefficient is a function of the temperature difference between the radiator and the air in the surrounding environment. As mentioned above, it is proportional to $\mathrm{Ra}^{1/3}$ or $\mathrm{Ra}^{1/4}$, depending on the Rayleigh number. Thus, it can be represented in the following combined form

$$\alpha\left(T - T_\mathrm{air}\right) = b\left(T - T_\mathrm{air}\right)^n, \tag{6}$$

with b as the proportionality thermal constant calculated from Eq. (3) with respect to Eq. (4). It is thus a complex function of the geometrical configuration of the radiator and the physical properties of the air in the environment. Commonly there is $n \approx 1.3$.

From Eq. (5), one then obtains

$$-c_\mathrm{p}\dot{m}\mathrm{d}T = b\left(T - T_\mathrm{air}\right)^n \mathrm{d}A\,. \tag{7}$$

Integration of this equation leads to the temperature distribution along the radiator surface

$$T - T_\mathrm{air} = \frac{1}{\left[\dfrac{1}{\left(T' - T_\mathrm{air}\right)^{n-1}} + (n-1)\dfrac{b}{c_\mathrm{p}\dot{m}}A_\mathrm{x}\right]^{1/(n-1)}}, \tag{8}$$

with T' as the temperature of the working fluid at the radiator inlet and A_x as the dependent heat transfer area along the radiator surface in the flow direction.

On the other hand, the total heat output is calculated from Eq. (5), taking into account Eq. (6), by the following integration:

$$\dot{Q} = \int_0^A \alpha\left(T - T_\mathrm{air}\right)\mathrm{d}A_\mathrm{x} = b\int_0^A \left(T - T_\mathrm{air}\right)^n \mathrm{d}A_\mathrm{x}\,. \tag{9}$$

Substituting Eq. (8), the integration finally leads to

$$\dot{Q} = c_\mathrm{p}\dot{m}\left(T' - T_\mathrm{air}\right)\left\{1 - \left[\frac{c_\mathrm{p}\dot{m}}{c_\mathrm{p}\dot{m} + (n-1)bA\left(T' - T_\mathrm{air}\right)^{n-1}}\right]^{\frac{1}{n-1}}\right\}. \tag{10}$$

This equation can also be found in [3] and in the given references, where the exponent n is defined differently. In place of the constant b, the symbol k was used in [3]. Unfortunately, no further analyses were conducted there, so that the calculation examples shown in [3] all refer to example radiators and therefore are not general for other types of radiators.

Obviously, Eq. (10) represents the precise form of Eq. (1). Not only the geometric parameter A of the radiator but also the thermal constant b was included. The calculation itself seems to be quite complex. Because the calculation refers to a single radiator, the similarity of thermal performance between different types of radiators cannot be used further. In addition, as mentioned in the introduction, the calculation does not refer to a known reference point and therefore does not account for other possible influencing factors such as the wall thickness of the radiator and the non-vanishing convective thermal resistance in the internal flow.

For this reason, Eq. (10) should be further developed.

2.2 Derivation of Main Equations

For further calculations, the thermal effectiveness of the radiator is first defined as follows

$$a = \frac{T' - T''}{T' - T_\mathrm{air}} = \frac{1}{T' - T_\mathrm{air}}\frac{\dot{Q}}{c_\mathrm{p}\dot{m}}, \tag{11}$$

where c_p is the specific heat capacity of the air and T'' is the exit temperature of the fluid flow from the radiator.

As stated in Eq. (2), it seems reasonable to always calculate the thermal characteristics of a given radiator from a known reference operating point. Such a reference point is basically specified by a given mass flow with known inlet and exit temperatures. The thermal output is then simply calculated as $\dot{Q}_\mathrm{R} = c_\mathrm{p}\dot{m}_\mathrm{R}\left(T'_\mathrm{R} - T''_\mathrm{R}\right)$. Accordingly, a_R is obtained from Eq. (11). For radiators, the reference point can be equated with the nominal point, which is specified in DIN EN442 to $T'_\mathrm{N} = 75$ °C for the inlet flow and $T''_\mathrm{N} = 65$ °C for the outlet flow of the radiator. With the nominal temperature of the surrounding air of $T_\mathrm{air,N}$=20 °C, one obtains $a_\mathrm{N} = 0.182$ as a special reference point.

The reason for using the term "reference point" instead of "design point" is that a radiator can theoretically be operated at all possible temperatures and flow rates. Thus, its design point is not defined before the heat power output for a particular application is specified.

Depending on the air and fluid flow temperatures, which may differ from the reference values, the heat output is calculated according to Eq. (11)

$$\frac{\dot{Q}}{\dot{Q}_\mathrm{R}} = \frac{a}{a_\mathrm{R}}\frac{\dot{m}}{\dot{m}_\mathrm{R}}\theta_\mathrm{inlet}, \tag{12}$$

with

$$\theta_\mathrm{inlet} = \frac{T' - T_\mathrm{air}}{T'_\mathrm{R} - T_\mathrm{air,R}}. \tag{13}$$

If a radiator operates only with mass flow adjustment, i.e., with an inlet temperature and an air temperature equal to those at the reference point, then there is simply $\theta_\mathrm{inlet} = 1$. In the extended sense, $\theta_\mathrm{inlet} = 1$ also includes the simultaneous change of T' and T_air under the condition $T' - T_\mathrm{air} = T'_\mathrm{R} - T_\mathrm{air,R}$.

Considering the parameter a defined in Eq. (11), then Eq. (10) is further written as

$$1 - a = \left[1 + (n-1)\frac{bA}{c_\mathrm{p}\dot{m}}\left(T' - T_\mathrm{air}\right)^{n-1}\right]^{\frac{-1}{n-1}}. \tag{14}$$

For the reference point this becomes

$$1 - a_\mathrm{R} = \left[1 + (n-1)\frac{bA}{c_\mathrm{p}\dot{m}_\mathrm{R}}\left(T'_\mathrm{R} - T_\mathrm{air,R}\right)^{n-1}\right]^{\frac{-1}{n-1}}. \tag{15}$$

With respect to Eq. (14) and the definition of θ_inlet in Eq. (13), the ratio of the heat outputs is obtained from Eq. (12)

$$\frac{\dot{Q}}{\dot{Q}_\mathrm{R}} = \frac{1}{a_\mathrm{R}}\frac{\dot{m}}{\dot{m}_\mathrm{R}}\theta_\mathrm{inlet} \cdot \left\{1 - \left[\frac{c_\mathrm{p}\dot{m}}{c_\mathrm{p}\dot{m} + \theta_\mathrm{inlet}^{n-1}(n-1)bA\left(T'_\mathrm{R} - T_\mathrm{air,R}\right)^{n-1}}\right]^{\frac{1}{n-1}}\right\}. \tag{16}$$

For further calculations, one first obtains from Eq. (15)

$$(n-1)bA\left(T'_\mathrm{R} - T_\mathrm{air,R}\right)^{n-1} = c_\mathrm{p}\dot{m}_\mathrm{R}\left(\frac{1}{\left(1-a_\mathrm{R}\right)^{n-1}} - 1\right). \tag{17}$$

This is inserted into Eq. (16). It then follows

$$\frac{\dot{Q}}{\dot{Q}_\mathrm{R}} = \frac{1}{a_\mathrm{R}}\frac{\dot{m}}{\dot{m}_\mathrm{R}}\theta_\mathrm{inlet} \cdot \left\{1 - \left[1 + \theta_\mathrm{inlet}^{n-1}\frac{\dot{m}_\mathrm{R}}{\dot{m}}\left(\frac{1}{\left(1-a_\mathrm{R}\right)^{n-1}} - 1\right)\right]^{\frac{-1}{n-1}}\right\}. \tag{18}$$

This equation is called the *first main equation* or the thermal characteristic equation of the radiator. In the general form it is written as

$$\frac{\dot{Q}}{\dot{Q}_\mathrm{R}} = \mathrm{f}\left(a_\mathrm{R}, \frac{\dot{m}}{\dot{m}_\mathrm{R}}, \theta_\mathrm{inlet}\right). \tag{19}$$

It exactly fulfills the expectation of Eq. (2). The thermal effectiveness of the radiator at the reference point (a_R) is a known quantity. All geometrical and air property parameters (β, ρ, ν, λ, c_p, Pr), including the thermal constant b in Eq. (6), do not need to be considered. Because of the use of the known reference point, Eq. (18) is also highly accurate.

Figure 1 shows the calculation results for a given radiator with known reference point $a_\mathrm{R} = 0.182$. It can be used to regulate the power output of the radiator by regulating either the mass flow rate or the inlet temperature (or both) of the working fluid.

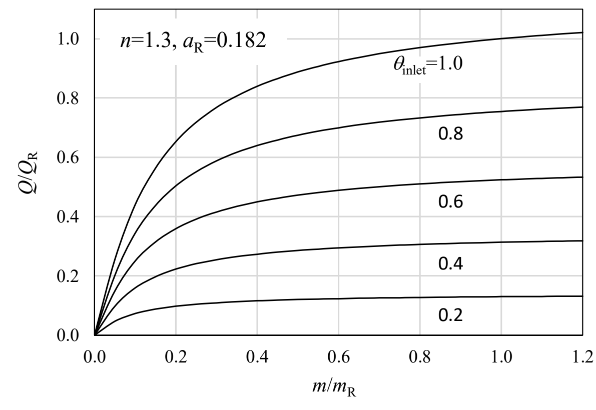

Fig. 1 Relative heat output of a radiator in dependence on mass flow rate and inlet temperature of the working medium

Comparing Eq. (18) with Eq. (12), one obtains the corresponding thermal effectiveness

$$a = 1 - \left[1 + \theta_\mathrm{inlet}^{n-1}\frac{\dot{m}_\mathrm{R}}{\dot{m}}\left(\frac{1}{\left(1-a_\mathrm{R}\right)^{n-1}} - 1\right)\right]^{\frac{-1}{n-1}}. \tag{20}$$

Furthermore, combining Eq. (18) and Eq. (12) for eliminating the term $\dot{m}/\dot{m}_\mathrm{R}$ yields

$$\frac{\dot{Q}}{\dot{Q}_R} = \frac{a}{a_R} \frac{\frac{1}{(1-a_R)^{n-1}} - 1}{\frac{1}{(1-a)^{n-1}} - 1} \theta_{inlet}^n . \tag{21}$$

This can be considered as another form of the *first main equation* of the radiator.

The exit temperature, which is contained in the parameter *a* according to Eq. (11), can be calculated by taking Eq. (12) into account as follows

$$\theta_{exit} = \frac{T'' - T_{air}}{T'_R - T_{air,R}} = \theta_{inlet} - a_R \frac{\dot{m}_R}{\dot{m}} \frac{\dot{Q}}{\dot{Q}_R} . \tag{22}$$

Alternatively, by substituting $\dot{Q}/\dot{Q}_R$ from Eq. (12), one also obtains

$$\theta_{exit} = (1-a)\theta_{inlet} . \tag{23}$$

Substituting Eq. (18) into Eq. (22), the *second main equation* of the radiator is obtained as

$$\theta_{exit} = \left[\frac{1}{\theta_{inlet}^{n-1}} + \frac{\dot{m}_R}{\dot{m}} \left(\frac{1}{(1-a_R)^{n-1}} - 1 \right) \right]^{\frac{-1}{n-1}} . \tag{24}$$

For the same radiator, as in **Fig. 1**, the calculation results of the exit temperature are shown in **Fig. 2**. Both Eq. (24) and **Fig. 2** are given here only for completeness of calculations. In practical applications of radiators, the exit temperature of the working medium can be used as a control or monitoring parameter.

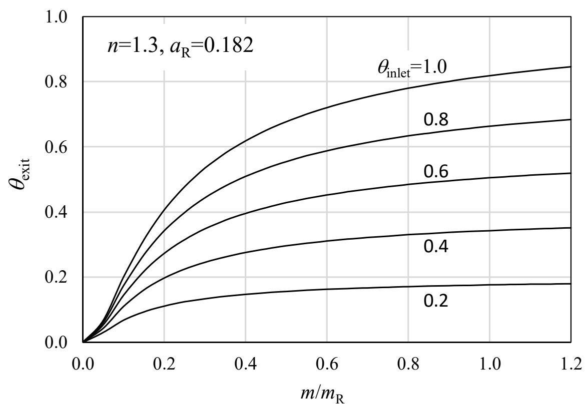

Fig. 2 Relative exit flow temperature of a radiator in dependence on mass flow rate and inlet temperature of the working medium

Furthermore, the temperature along the radiator surface in the direction of flow, as given in Eq. (8), can be calculated by combining Eq. (14), leading to

$$T^* = \frac{T - T_{air}}{T' - T_{air}} = \left[1 + \left(\frac{1}{(1-a)^{n-1}} - 1 \right) \frac{A_x}{A} \right]^{\frac{-1}{n-1}} . \tag{25}$$

The temperature is related to the temperature at the inlet of the radiator. The calculated temperatures as a function of the running surface A_x/A and for different values of the effectiveness of the radiator are shown in **Fig. 3**, where n=1.3 was assumed.

For $A_x/A = 1$, one obtains the exit temperature $T'' - T_{air} = (1-a)(T' - T_{air})$, which agrees with Eq. (23).

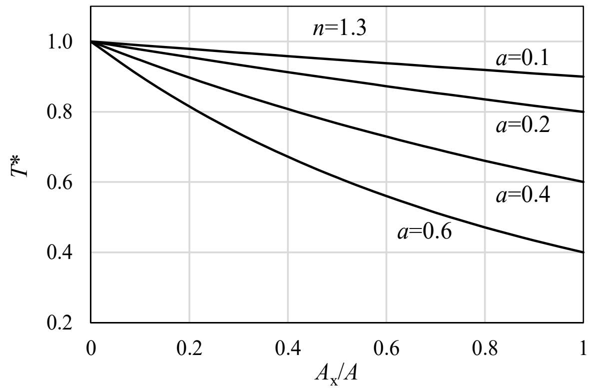

Fig. 3 Temperature on the surface of the radiator

For operation under the nominal condition (i.e. $a_R = a_N = 0.182$) and with n=1.3, it follows

$$T_N^* = \left(1 + 0.062 \frac{A_x}{A} \right)^{\frac{-1}{0.3}} . \tag{26}$$

2.3 Special Cases

For the mass flow of the working medium equal to that at the reference point, it follows from Eq. (18) with $\dot{m} = \dot{m}_R$:

$$\frac{\dot{Q}}{\dot{Q}_R} = \frac{1}{a_R} \theta_{inlet} \left\{ 1 - \left[1 + \theta_{inlet}^{n-1} \left(\frac{1}{(1-a_R)^{n-1}} - 1 \right) \right]^{\frac{-1}{n-1}} \right\} . \tag{27}$$

The power output is now only a function of the inlet flow temperature θ_{inlet} (with a_R as a known constant). The case $a_R = 1$ means that the exit temperature of the working fluid equals the temperature of the ambient air. This further means that, for a given radiator, the reference mass flow rate is sufficiently low, as can be confirmed from Eq. (15) with $\dot{m}_R \to 0$. From Eq. (27), it then follows $\dot{Q}/\dot{Q}_R = \theta_{inlet}$. The power output is proportional to the inlet temperature of the working fluid.

Figure 4 shows the calculation results of Eq. (27). It can be applied to regulate the radiator output.

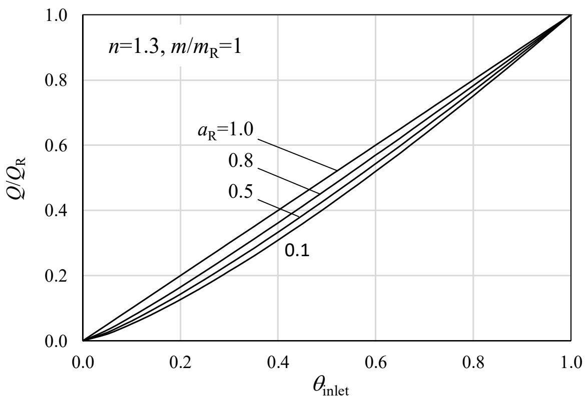

Fig. 4 Relative heat output of radiators with same mass flow as in reference points

The exit temperature given in Eq. (24) is accordingly simplified to

$$\theta_{\text{exit}} = \left(\frac{1}{\theta_{\text{inlet}}^{n-1}} + \frac{1}{\left(1 - a_{\text{R}}\right)^{n-1}} - 1 \right)^{\frac{-1}{n-1}} . \tag{28}$$

Corresponding calculation results are shown in **Fig. 5**. For $a_{\text{R}} = 1$ which also means a=1 according to Eq. (20), $\theta_{\text{exit}} = 0$ results from Eq. (23) automatically.

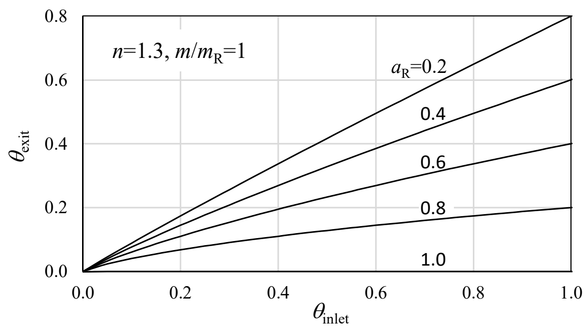

Fig. 5 Relative exit temperatures of radiators with same mass flow as in reference points

Another special case is given for constant inlet flow temperature. With $\theta_{\text{inlet}} = 1$, Eq. (18) is simplified to

$$\frac{\dot{Q}}{\dot{Q}_{\text{R}}} = \frac{1}{a_{\text{R}}} \frac{\dot{m}}{\dot{m}_{\text{R}}} \left\{ 1 - \left[1 + \frac{\dot{m}_{\text{R}}}{\dot{m}} \left(\frac{1}{\left(1 - a_{\text{R}}\right)^{n-1}} - 1 \right) \right]^{\frac{-1}{n-1}} \right\} . \tag{29}$$

The expression in the curly brackets is, according to Eq. (20), the corresponding thermal effectiveness of the radiator operating at $\theta_{\text{inlet}} = 1$.

In this case, the heat output is a function of the mass flow of the working fluid for each given a_{R} (reference operating point of the radiator), as shown in **Fig. 6**. The fact to be mentioned is that this type of diagram can be found in many literatures and textbooks, but not as accurate as calculated here for the first time based on Eq. (29). In [**3**], similar calculations are shown for a given radiator (n=1.3, k=3.092, A=2.041) based on Eq. (10) for different reference values of power outputs varying from 125 W to 1000 W, which is achieved by varying the inlet temperature. The examples are not non-dimensional. By remapping this varying output power to varying effectiveness a_{R}, as we did, the calculation examples in [**3**] were recalculated and verified by fully satisfying Eq. (29) and **Fig. 6** in dimensionless form.

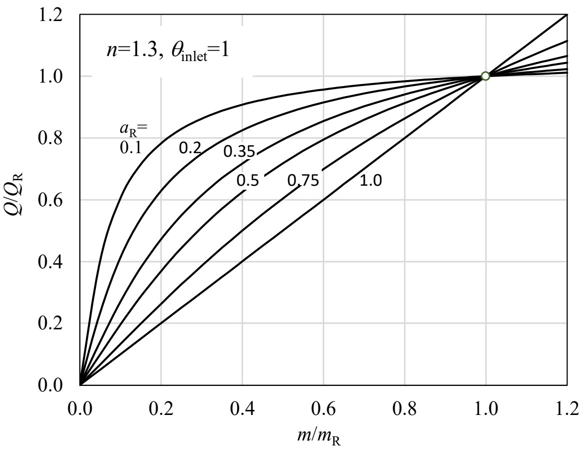

Fig. 6 Heat output of radiators with same inlet temperatures as in reference points

The thermal characteristics shown in **Fig. 6** are valid both for different radiators (i) and for a given radiator (ii) with different reference operating points. In the former case (i), the diagram is commonly applicable to radiator selection in connection with installation and load adjustment in building services engineering, i.e., HVAC applications [**6, 7**]. Together with an equal-percentage control valve, for instance, the approximate linear adjustment of the power output can be achieved. In the latter case (ii), different curves correspond to different reference points ($\dot{m}_{\text{R}}$, T'_{R}, $T_{\text{air,R}}$) of a given radiator. For each curve, however, the relationship between the mass flow rate and the power output is subject to the condition $\theta_{\text{inlet}} = 1$. This condition, unfortunately, has not always been clearly stated in the past. For this reason, the general case of Eq. (18), and hence Eq. (29), is much more advanced than all previous calculations.

In practical applications, for instance in **[6]** and in many other German textbooks, the approximation in place of Eq. (29) has often been found as follows:

$$\frac{\dot{Q}_{\text{app}}}{\dot{Q}_{\text{R}}} \approx \frac{1}{1 + a_{\text{R}}\left(\frac{\dot{m}_{\text{R}}}{\dot{m}} - 1\right)}. \quad (30)$$

In contrast to Eq. (29), the influence of the exponent n is not taken into account.

When denoting Eq. (29) as the accurate calculation ($\dot{Q}_{\text{acc}}$), the ratio of $\dot{Q}_{\text{app}}/\dot{Q}_{\text{acc}}$, which is the relative accuracy of Eq. (30), can be calculated, as shown in **Fig. 7**. The approximation according to Eq. (30) is obviously inaccurate for low flow rates with, say, $\dot{m}/\dot{m}_{\text{R}} < 0.5$. Such an inaccuracy depends on the thermal effectiveness of the radiator (a_{R}) as well as on the exponent n, which is found in $\dot{Q}_{\text{acc}}$ in Eq. (29), but not in Eq. (30).

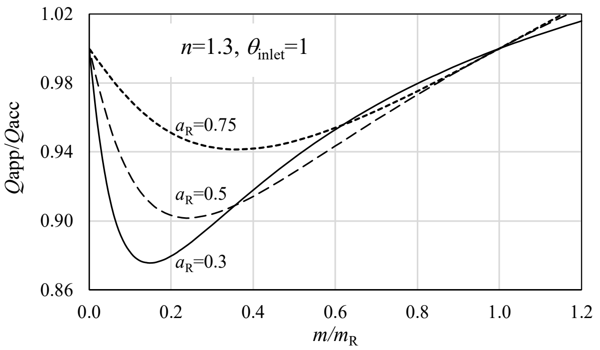

Fig. 7 Relative accuracy of the approach equation which has been used in practice

The further outcome of the special case for $\theta_{\text{inlet}} = 1$, correspondingly, is the dimensionless exit temperature, as obtained from Eq. (24)

$$\theta_{\text{exit}} = \left[1 + \frac{\dot{m}_{\text{R}}}{\dot{m}}\left(\frac{1}{(1-a_{\text{R}})^{n-1}} - 1\right)\right]^{\frac{-1}{n-1}}, \quad (31)$$

as shown in **Fig. 8**.

This diagram, as in **Fig. 6**, is again valid for different radiators as well as for a specific radiator with different reference operating points. As a matter of fact, such a diagram for the exit temperature of radiators, like that of **Fig. 2**, has been missing in all applications up to now. Equations (24) and (31) therefore represent highly advanced calculations. They deserve, together with Eqs. (18) and (29) which will find more application below in section 4, the name of the *main equations* of radiators.

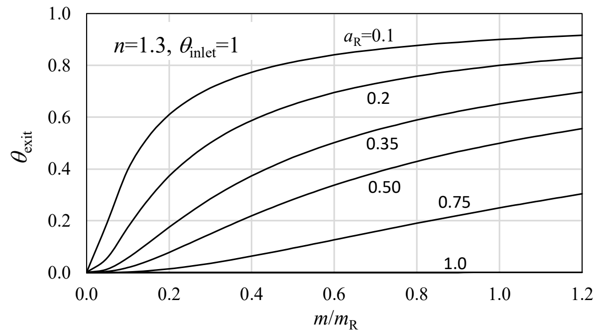

Fig. 8 Related exit temperatures of radiators with same inlet temperatures as in reference points

2.4 Reverse Calculations

In certain applications of radiators, it may be required to control the mass flow rate or the inlet temperature of the working medium, in order to achieve a desired heat output or a desired exit temperature. This just represents a practical application example of the thermal characteristics (diagrams) of radiators. Basically, the desired operation of the radiator can be directly read out from the corresponding diagram such as **Fig. 6** for a given power output or **Fig. 8** for a given exit temperature. However, for the realization of an automatic control, it seems very convenient if the corresponding calculations can be directly implemented in the software used.

In this case, one needs to use the main equations viz. Eqs. (18) and (24). From Eq. (24), for instance, the necessary mass flow for a given inlet temperature and a desired exit temperature can be directly calculated as follows

$$\frac{\dot{m}}{\dot{m}_{\text{R}}} = \theta_{\text{inlet}}^{n-1} \frac{\frac{1}{(1-a_{\text{R}})^{n-1}} - 1}{(\theta_{\text{inlet}}/\theta_{\text{exit}})^{n-1} - 1}. \quad (32)$$

Correspondingly, for a given mass flow and a desired exit flow temperature, the necessary inlet temperature is simply obtained from the above equation

$$\theta_{\text{inlet}} = \left[\frac{1}{\theta_{\text{exit}}^{n-1}} - \frac{\dot{m}_{\text{R}}}{\dot{m}}\left(\frac{1}{(1-a_{\text{R}})^{n-1}} - 1\right)\right]^{\frac{-1}{n-1}}. \quad (33)$$

If a desired power output $\dot{Q}/\dot{Q}_{\text{R}}$ should be reached, then, as can be confirmed from Eq. (18), either the inlet temperature or the mass flow rate can be regulated. In the former case, the inlet temperature is solved iteratively from Eq. (27) for $\dot{m} = \dot{m}_{\text{R}}$. In the latter case, iterative calculations of Eq. (29) for $\theta_{\text{inlet}} = 1$ must be completed.

3 Review of the Approach in DIN EN442

As shown above, Eq. (18) for calculating the thermal characteristics of radiators by using the reference point is much more convenient than Eq. (10) for individual calculations. This concept, in fact, was also applied in DIN EN442. Instead of using the reference mass flow, however, a nominal temperature difference ΔT_{N} is specified as the reference value. The ratio of the heat output of a radiator to the nominal value $\dot{Q}_{\mathrm{N}}$ is then approached by

$$\frac{\dot{Q}}{\dot{Q}_{\mathrm{N}}}=\left(\frac{\Delta T}{\Delta T_{\mathrm{N}}}\right)^{n}. \tag{34}$$

The exponent n, usually n=1.3, has its background in connection with the definitions of the Nusselt and Rayleigh numbers as well as with Eq. (6).

For the hypothesis of Eq. (34), see also VDI Heat Atlas [**4**].

The nominal temperature difference is obtained from the nominal temperatures, which are set at $T'_{\mathrm{N}}=75$ °C for the inlet flow and $T''_{\mathrm{N}}=65$ °C for the exit flow from the radiator. With a nominal temperature of the ambient air at $T_{\mathrm{air,N}}$=20 °C, the nominal logarithmic temperature difference is calculated as $\Delta T_{\mathrm{N,log}}=49.8$ °C. The arithmetic mean of the temperature difference is $\Delta T_{\mathrm{N,arith}}=50$ °C. The effectiveness of the radiator, as already obtained from Eq. (11), is $a_{\mathrm{N}}=0.182$.

Obviously, Eq. (34) stands for the thermal similarity between all types of radiators. The use of the temperature difference, however, is less reasonable than the use of the mass flow rate according to Eq. (19), because the temperature difference is lastly determined by the mass flow rate. In addition, Eq. (34) is obtained based on the use of a mean heat transfer coefficient α which in fact changes along the radiator surface in the flow direction, see Eq. (6). Thus, Eq. (34) is only an approximation even when using the logarithmic temperature difference. Another fact worth mentioning is that it is still unresolved how to estimate the error associated with the use of Eq. (34). In [**3**], for instance, it states that based on experimental testing the logarithmic temperature difference is the "closest to reality".

Based on the derivation of the two main equations in Sect. 2.2, the error associated with Eq. (34) can be accurately determined as follows.

Taking as reference the nominal point specified by $a_{\mathrm{R}}=a_{\mathrm{N}}=0.182$ and $\dot{Q}_{\mathrm{R}}=\dot{Q}_{\mathrm{N}}$, it follows from Eq. (21) with n=1.3:

$$\frac{\dot{Q}}{\dot{Q}_{\mathrm{N}}}=\frac{0.341a}{\dfrac{1}{(1-a)^{0.3}}-1}\theta_{\mathrm{inlet}}^{1.3}. \tag{35}$$

Its comparison with Eq. (34) provides information about the inaccuracy of Eq. (34). First, it is assumed that the logarithmic temperature difference is used in Eq. (34), which is then rewritten as follows

$$\frac{\dot{Q}_{\mathrm{DIN,\,log}}}{\dot{Q}_{\mathrm{N}}}=\left(\frac{1}{49.83}\frac{T'-T''}{\ln\dfrac{T'-T_{\mathrm{air}}}{T''-T_{\mathrm{air}}}}\right)^{n}. \tag{36}$$

Taking into account Eq. (13) for the definition of θ_{inlet} as well as Eq. (22) for θ_{exit}, one obtains with $T'_{\mathrm{N}}-T_{\mathrm{air,N}}=55$

$$\frac{\dot{Q}_{\mathrm{DIN,\,log}}}{\dot{Q}_{\mathrm{N}}}=\left(\frac{55}{49.83}\frac{\theta_{\mathrm{inlet}}-\theta_{\mathrm{exit}}}{\ln\dfrac{\theta_{\mathrm{inlet}}}{\theta_{\mathrm{exit}}}}\right)^{1.3}. \tag{37}$$

With respect to Eq. (23), one further obtains

$$\frac{\dot{Q}_{\mathrm{DIN,\,log}}}{\dot{Q}_{\mathrm{N}}}=1.137\frac{a^{1.3}\theta_{\mathrm{inlet}}^{1.3}}{\left(\ln\dfrac{1}{1-a}\right)^{1.3}}. \tag{38}$$

The ratio of Eq. (38) to Eq. (35) is built as

$$\frac{\dot{Q}_{\mathrm{DIN,\,log}}}{\dot{Q}}=\frac{3.33a^{0.3}}{\left(\ln\dfrac{1}{1-a}\right)^{1.3}}\left[\frac{1}{(1-a)^{0.3}}-1\right]. \tag{39}$$

It is expressed as a function of the effectiveness a of the radiator.

Similarly, for the use of the arithmetic temperature difference in Eq. (34), one obtains

$$\frac{\dot{Q}_{\mathrm{DIN,arith}}}{\dot{Q}}=1.347\frac{(2-a)^{1.3}}{a}\left[\frac{1}{(1-a)^{0.3}}-1\right]. \tag{40}$$

Both Eq. (39) and Eq. (40) are presented in **Fig. 9**. Obviously, Eq. (34) from DIN EN442 is simply a rough approximation even when using the logarithmic temperature difference. The use of Eq. (34) with logarithmic temperature difference leads to an overestimation of the heat output and, when designing radiators, to an underestimation of the radiator size. For a reliable application of Eq. (34) with an inaccuracy of less than 1%, the logarithmic temperature difference should be restricted to the range of about *a*<0.55. The arithmetic temperature difference in Eq. (34) should generally not be used.

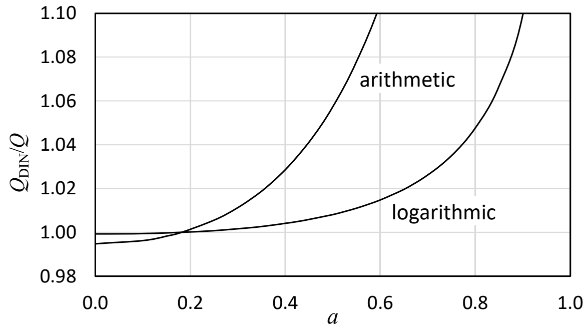

Fig. 9 Deviations of the approach of Eq. (34) with logarithmic and arithmetic temperature difference from real values of heat output

At the nominal point $a = a_N = 0.182$, as being expected, there are $\dot{Q}_{DIN,log}/\dot{Q} = 1$ and $\dot{Q}_{DIN,arith}/\dot{Q} = 1$.

As can be verified, Eqs. (39) and (40) satisfy the general relationship:

$$\frac{\Delta T_{arith}}{\Delta T_m} = \left(\frac{1}{a} - \frac{1}{2}\right) \ln \frac{1}{1-a}. \tag{41}$$

This relation is shown in **Fig. 10**. The approximation with the arithmetic mean of the temperature difference is only sufficient for small values of thermal effectiveness. According to VDI Heat Atlas [**4**], the condition for using the arithmetic temperature difference is restricted to $(T'' - T_{air})/(T' - T_{air}) > 0.7$ i.e. $a < 0.3$. When applied to radiators, as in **Fig. 9**, a maximum error of 1.1% could occur.

Because of Eq. (41), the thermal effectiveness of a heat exchanger also behaves as a scale parameter for assessing the applicability of the arithmetic temperature difference in place of the logarithmic one. Note that, according to **Fig. 9**, neither the logarithmic nor the arithmetic temperature difference is accurate for radiators.

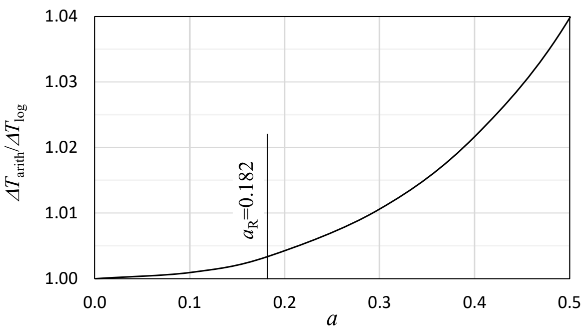

Fig. 10 Ratio of arithmetic to logarithmic temperature differences as a function of the effectiveness of the radiator.

4 Eigen-Constant and Eigen-Parameter of Radiators

For radiators, the thermal constant *b* in Eq. (6) can be determined from Eqs. (3) and (4):

$$b = C\left(\frac{g\beta}{\nu^2}\mathrm{Pr}\right)^{1/3} \lambda . \tag{42}$$

Under the nominal condition, which is specified by p_{air}=1 bar and $T_{air,N} = 20$ °C, the following quantities of air are found:

$\beta = 3.425 \cdot 10^{-3}$ 1/K,
$\nu = 1.536 \cdot 10^{-5}$ m²/s,
$\lambda = 25.68 \cdot 10^{-3}$ W/Km,
$\mathrm{Pr} = 0.715$.

Then one obtains from Eq. (42)

$$b_N = 12C . \tag{43}$$

The significance of Eq. (42) or Eq. (43) for $T_{air,N} = 20$ lies in the direct determination of the free convection constant *C* from the thermal constant *b* or b_N based on experiments.

First, one obtains from Eq. (17)

$$\Omega = \frac{bA}{c_p \dot{m}_R} = \frac{1}{n-1} \frac{1}{\left(T'_R - T_{air,R}\right)^{n-1}} \left[\frac{1}{\left(1-a_R\right)^{n-1}} - 1\right], \tag{44}$$

which is denoted as the *eigen-constant* of radiators.

Under the nominal condition as in Sect. 3, for instance, it follows with *n*=1.3 and $a_N = 0.182$

$$\Omega_N = \frac{b_N A}{c_p \dot{m}_N} = 0.0622 . \tag{45}$$

The calculation of the *eigen-constant* of radiators is comparable with NTU, for instance, for plate heat exchangers. Since in radiators the heat transfer coefficient on the radiator surface is not constant in the direction of flow, the use of NTU is irrelevant.

As can be seen from the above equation, it is not the surface area of the radiator alone, but always the combination of $b_N A$ that determines the radiator design and calculations. It is comparable with *kA* for plate heat exchangers (with *k* as the heat transfer coefficient). The significance of this property of radiators is that one does not have to specify the exact surface area of the radiator, which is often not clearly defined, e.g. in the case of wavy surfaces: one may use either the real surface area or the projection area. Furthermore, the constant b_N is connected with the constant *C* by Eq. (43). The latter depends on the surface structure, surface orientation, edge effect, etc. Only rough values for flat surfaces can be found in literature. For this reason and because of its importance, the term $E=bA$ or $E_N = b_N A$ is denoted as the *eigen-parameter* of radiators. Its ratio to the heat capacity rate $c_p \dot{m}_N$ is the *eigen-constant* of radiators.

4.1 Experimental Methods

Equation (44) or Eq. (45) also provides a convenient method for experimentally determining the thermal constant b or b_N and further constant C in Eq. (3) according to Eq. (42) and (43), respectively. In the experiment, one measures the outlet temperature at a given inlet temperature and then calculates the thermal effectiveness a according to Eq. (11). Then, the *eigen-constant* of the radiator can be determined from Eq. (44). With the measured mass flow and a selected reference area A, the constant b is further obtained. The free convection constant C is then obtainable from Eq. (42) or Eq. (43) for $T_{\mathrm{air,N}} = 20$ °C.

The experiment can also be performed as follows: The radiator is found in an atmosphere with a temperature of $T_{\mathrm{air,N}} = 20$ °C. At the nominal inlet temperature $T_N' = 75$ °C, the mass flow is to be regulated until the nominal exit temperature of $T_N'' = 65$ °C is reached. With the measured mass flow and a selected reference area A, the thermal constant b_N can be determined from Eq. (45). Then, the constant C is further determined from Eq. (43). Finally, the Nusselt number and the heat transfer coefficient is obtainable from Eq. (3).

4.2 Application of Eigen-Constant

Another important application of the radiator *eigen-constant* is briefly presented here. The application is found where, for the same mass flow and the same inlet temperature, the power output, e.g., should be increased/decreased by a factor k_Q by serially enlarging/reducing the radiator surface. According to Eq. (11), the change in power output is proportional to the change in thermal effectiveness, so that $a_{II} = k_Q a_I$ exists. The necessary enlargement/ reduction in radiator surface can then be calculated from Eq. (44) as follows:

$$\frac{A_{II}}{A_I} = \frac{\Omega_{II}}{\Omega_I} = \frac{\dfrac{1}{\left(1-k_Q a_I\right)^{n-1}} - 1}{\dfrac{1}{\left(1-a_I\right)^{n-1}} - 1} . \tag{46}$$

For k_Q=2, for instance, the thermal effectiveness of the radiator is twice the initial value. The required surface, as calculated from the above equation, is more than twice as large, depending on the initial value of effectiveness a_I (for a_I=0.3, then, A_{II}/A_I=2.8). The same computational concept applies to the series connection of two radiators of the same design.

On the other hand, Eq. (46) can also be used to determine the change in power output as a function of the change in radiator area (height). One obtains

$$\frac{\dot{Q}_{II}}{\dot{Q}_I} = k_Q = \frac{1}{a_I} - \frac{1}{a_I}\left\{\frac{A_{II}}{A_I}\left[\frac{1}{\left(1-a_I\right)^{n-1}} - 1\right] + 1\right\}^{\frac{-1}{n-1}} . \tag{47}$$

This equation, like Eq. (21), can also be considered as another form of the *first main equation* of the radiator. It represents the ratio of the thermal performances of two radiators that have the same design but different surface areas (or heights) and operate at the same mass flow rate and the same inlet temperature.

Another significant value of Eq. (44) can be confirmed. For two radiators (I and II) of different size but with the equal thermal effectiveness ($a_{R,I} = a_{R,II}$), Eq. (44) yields

$$\frac{A_{II}}{A_I} = \frac{b_I}{b_{II}} \frac{c_{p,II}\dot{m}_{R,II}}{c_{p,I}\dot{m}_{R,I}} \left(\frac{T_{R,I}' - T_{\mathrm{air},R,I}}{T_{R,II}' - T_{\mathrm{air},R,II}}\right)^{n-1} . \tag{48}$$

For the same inlet temperature and the same air temperature, as well as for the same thermal constant b and the same heat capacity c_p, it follows further

$$\frac{A_{II}}{A_I} = \frac{\dot{m}_{R,II}}{\dot{m}_{R,I}} . \tag{49}$$

Because of $\dot{Q} = c_p \dot{m}\left(T' - T''\right)$, one further obtains

$$\frac{\dot{Q}_{II}}{\dot{Q}_I} = \frac{A_{II}}{A_I} . \tag{50}$$

As might have been expected, the power output is simply proportional to the surface area of the radiators. This relationship represents the exact thermal similarity between radiators with similar geometries but of different sizes. It differs from Eq. (47), which is found for two differently sized radiators with the same mass flow rate.

A comparable thermal similarity can also be found in plate heat exchangers [**8**].

The above equation can be verified by checking the linearities of known thermal data of radiators, which can be found, for instance, in the VDI Heat Atlas [**4**]. To this end, three different arrangements of plate radiators are considered according to **Fig. 11**.

For each installation form, the power output is given as a function of the radiator height (BH) between 0.2 and 1 m. With the width of the plate radiator equal to 1 m, the power output is given in kW/m. Thermal data are available for an inlet temperature of $T' = 90$ °C, an outlet temperature $T'' = 70$ °C, and an air temperature $T_{air} = 20$ °C. The thermal effectiveness is calculated as $a_1 = 0.286$. This constant value can only be given by using a variable mass flow that satisfies Eq. (49). It is then expected that Eq. (50) is fulfilled.

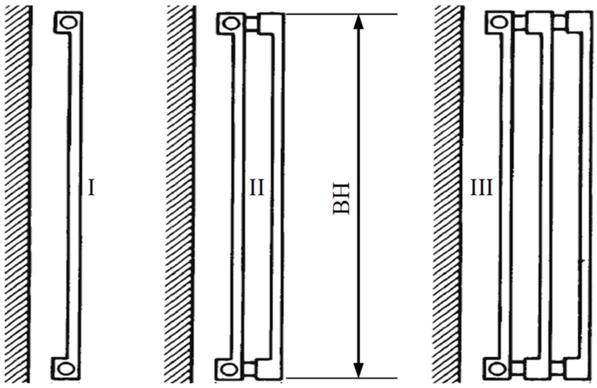

Fig. 11 Plate radiators and installations, from the VDI Heat Atlas [**4**]

Figure 12 shows the comparison between the available data and the expectations according to Eq. (50). The expected straight lines are in fact regression lines passing through the coordinate origin. For all three installation forms considered, the expectations based on Eq. (50) proved to be accurate, particularly for case I. This in turn reflects the high accuracy of the derived *main equation* for radiators given by Eq. (18). Finally, Eq. (3) is indirectly validated.

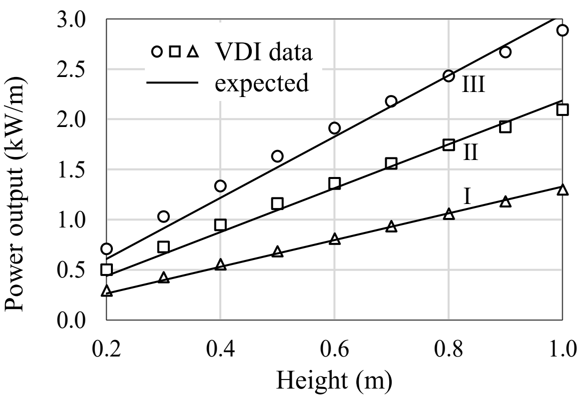

Fig. 12 Comparison between the available data (VDI Heat Atlas) and the expectations

5 Summary

For common reasons, the thermal characteristics of radiators, as in HVAC applications, could only be measured or estimated on the basis of empirical relationships, as found in the European standard DIN EN442. The studies in the present paper therefore focus on calculation methods of radiators. The key point is the use of a reference point, which is supposed to be known and is not necessarily the design or nominal point of the radiator considered. The most significant advantage of this method is that all the geometrical parameters of the radiator considered and the properties of the air in free convection are no longer needed for the calculations.

The accurate analyses are based on the consideration of the real variation of the heat transfer coefficient along the radiator surface, instead of assuming a constant. Two main equations, i.e., thermal characteristic equations of the radiator have been established, which allow all other operating points to be calculated simply and directly from the reference point. Because of their powerful effectiveness, the main equations can also be called the master equations. For performing special load adjustments of radiators, reverse calculations for different control concepts were also presented. Furthermore, the empirical relationship, i.e., the approach of the heat output of radiators found in DIN EN442 for buildings was reviewed for its accuracy and application restrictions. As was confirmed with the aid of the main equations, even the use of the logarithmic temperature difference in the given formula is inaccurate. For the design and calculation of radiators, both the eigen-constant and the eigen-parameter were introduced. Then, the experimental method for easily determining the heat transfer coefficient based on the use of eigen-constant was outlined. Furthermore, the introduced eigen-constant of radiators also contributes to the calculation of the power output as a function of the radiator area. Using the data available in the VDI Heat Atlas, corresponding applications and application accuracy were demonstrated.

Nomenclature

a	Thermal effectiveness
A	Area of radiator surface
b	Constant
C	Constant
c_p	Specific heat capacity of fluid
E	Eigen-parameter of the radiator
l	Characteristic length
$\dot{m}$	Mass flow rate
n	Exponent
Nu	Nusselt number
Pr	Prandtl number
$\dot{Q}$	Heat power output

Ra	Rayleigh number
T	Temperature
α	Heat transfer coefficient
β	Thermal expansion coefficient
θ	Dimensionless temperature difference
λ	Thermal conductivity of air
ν	Kinematic viscosity
Ω	Eigen-constant of the radiator

References

1. Kays W. and London A. (1964) Compact Heat Exchangers. 2nd ed., McGraw-Hill, New York, NY, USA
2. Zohuri B. (2017) Compact Heat Exchangers, Selection, Application, Design and Evaluation. Springer Verlag
3. Muniak D. (2017) Radiators in Hydronic Heating Installations: Structure, Selection and Thermal Characteristics. Springer Verlag, Cham, Switzerland
4. VDI (2010) VDI Heat Atlas, Springer Verlag
5. Baehr H. and Stephan K. (2008) Wärme- und Stoffübertragung. Springer Verlag, 6. Auflage
6. Betschart W. (2015) Hydraulik in der Gebäudetechnik. Faktor Verlag, Zürich
7. Schmid C. et al. (2020) Heizung, Lüftung, Elektrizität: Energietechnik im Gebäude. 6. Auflage, vdf Verlag
8. Zhang Zh., Schiess C. and Tillenkamp F. (2022) Main Equations and Thermal Characteristics of Plate Heat Exchangers. IEFE Energy Papers, 11 (2022), Nr. 2, DOI: 10.21256/zhaw-2535