Investigation of the Ramsey-Pierce-Bowman Model

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Abstract. In this publication, the model by Ramsey, Pierce, and Bowman for finding the hierarchical order of the various sectors of an economy, conceiving each of them as users or suppliers for other sectors of the economy, is investigated. Computational results for a benchmark instance are provided.

Keywords: Simulated Annealing · Threshold Accepting · Ramsey · Spin Glass · Traveling Salesman Problem

1 Introduction

In 1969, Ramsey, Pierce, and Bowman considered the problem to find a hierarchical order of the various sectors of an economy, conceiving each of them as users or suppliers of goods and services for other sectors, including itself [1]. For this purpose, an exchange matrix J between the various sectors is considered with $J(i,j) \geq 0$ being the value of all products of sector i which are used for manufacturing products of a specified normalized value in sector j. Examples of these sectors are banks and insurance companies, the automobile industry, the chemical industry, publishing companies, agriculture and fishing, transport, restaurants, but also non-profit organizations like churches, and the entertainment industry. The exchange matrix is generally asymmetric for data from real-world economies.

The problem the Ramsey-Pierce-Bowman (RPB) model treats is to find a hierarchical order σ of N economic sectors in the way that $\sigma(1)$ is the sector buying products of the largest value from other sectors, whereas $\sigma(N)$ is the sector providing the largest-valued supply of products to other sectors. Thus, $\sigma(1)$ is called the largest user, $\sigma(N)$ the largest supplier. To solve this problem, a cost function or Hamiltonian to be minimized is defined in [1]:

$$\mathcal{H}_J(\sigma) = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} J(\sigma(i), \sigma(j)) = \sum_{i<j} J(\sigma(i), \sigma(j)). \tag{1}$$

This Hamiltonian exhibits some very interesting properties [1]:

- If J is a symmetric matrix, then $\mathcal{H}_J \equiv$ const, as for each permutation σ, either $\sigma(i) < \sigma(j)$ or $\sigma(i) > \sigma(j)$ for $i \neq j$. Thus, either $J(\sigma(i), \sigma(j))$ or $J(\sigma(j), \sigma(i))$ is part of the upper triangular matrix and adds to the total costs $\mathcal{H}_J(\sigma)$ of the configuration σ. If now $J(\sigma(i), \sigma(j)) = J(\sigma(j), \sigma(i))$ for a pair (i, j), it does not make any difference which of these two values is added to $\mathcal{H}_J(\sigma)$.
- Furthermore, the Hamiltonian $\mathcal{H}_J$ is additive with respect to the underlying exchange matrix J: let $J = K + L$, then

$$\begin{aligned} \mathcal{H}_{K+L}(\sigma) &= \mathcal{H}_J(\sigma) \\ &= \sum_{i<j} J(\sigma(i), \sigma(j)) \\ &= \sum_{i<j} \left(K(\sigma(i), \sigma(j)) + L(\sigma(i), \sigma(j))\right) \\ &= \sum_{i<j} K(\sigma(i), \sigma(j)) + \sum_{i<j} L(\sigma(i), \sigma(j)) \\ &= \mathcal{H}_K(\sigma) + \mathcal{H}_L(\sigma). \end{aligned} \tag{2}$$

Combining these two properties, the aim of the optimization process can be described more precisely: let

$$K(i, j) = \min\{J(i, j), J(j, i)\} \tag{3}$$

and

$$L(i, j) = J(i, j) - K(i, j). \tag{4}$$

As K is symmetric, $\mathcal{H}_K \equiv$ const is a flow amount independent of the hierarchical order of the economic sectors in the exchange matrix and thus plays no role in the optimization process. Please note that the original optimization problem to minimize $\mathcal{H}_J$ is thus equivalent to the minimization of $\mathcal{H}_L$, in which only the differences between the flows within each pair (i, j) of economic sectors are considered.

The RPB model is of great interest as it is related to both infinite-dimensional spin glass models, like the Sherrington-Kirkpatrick (SK) model [2], and to the Traveling Salesman Problem (TSP) [3, 4]. The Hamiltonian of the SK model and related models is given by

$$\mathcal{H}_J(\sigma) = -\sum_{i<j} J(i, j) S_i S_j \tag{5}$$

with the configuration σ being comprised of N spins S_i, usually taking the values ± 1. Here J is a symmetric matrix with $J(i,j) = J(j,i)$ describing the interaction between the spins S_i and S_j. The optimization problem consists of finding optimum spin values leading to a minimum energy value. The traveling salesman has the task to find that permutation σ, for which the Hamiltonian

$$\mathcal{H}_J(\sigma) = J(\sigma(N), \sigma(1)) + \sum_{i=1}^{N-1} J(\sigma(i), \sigma(i+1)) \tag{6}$$

becomes minimal. Here $J(i,j)$ denotes the distance between two nodes i and j, usually measured in units of either length or time. Thus, the RPB model is right in the middle between the spin glass models and the TSP: like in the SK model, all entries of the upper triangular part of the exchange matrix are added to the cost function value, but one tries to find the optimum permutation for minimizing the energy like for the TSP.

For the TSP, Kobe and Klotz introduced a frustration measure m [5] which they called misfit parameter and which is defined as

$$m = \frac{\mathcal{H}_J(\sigma_0) - \mathcal{H}_J^{\mathrm{id}}}{\mathcal{H}_J^{\mathrm{id}}} \tag{7}$$

with σ_0 being the optimum configuration and $\mathcal{H}_J^{\mathrm{id}}$ being the cost function of an idealized unfrustrated system in which each node i can be connected to its two nearest neighbors $n_1(i)$ and $n_2(i)$, i.e.,

$$\mathcal{H}_J^{\mathrm{id}} = \frac{1}{2} \sum_{i=1}^{N} J(n_1(i), i) + J(i, n_2(i)). \tag{8}$$

The larger the misfit, the larger the deviation from this idealized trivial problem and thus the larger the frustration. A similar measure can be defined for the RPB problem as

$$m = \frac{\mathcal{H}_J(\sigma_0) - \mathcal{H}_K}{\mathcal{H}_K}, \tag{9}$$

with σ_0 being the optimum hierarchy and $\mathcal{H}_K$ being the constant energy value of the symmetrized trivial problem.

2 Other Ways of Defining a Hierarchical Order

The question arises why such a complex optimization problem has to be proposed for finding the hierarchical order of the various sectors of an economy and whether this problem definition could not be replaced by a much simpler approach, which introduces classification figures for all sectors, sorts these numbers according to their sizes, and leads to a result identical to the order one gets after an exact optimization of the RPB problem. There are various ways of how to define such classification figures, which shall be illustrated with a small, randomly created toy example. For this small instance, let $N = 4$ be the number of sectors and

$$J = \begin{pmatrix} 6 & 7 & 2 & 5 \\ 8 & 8 & 7 & 1 \\ 1 & 7 & 8 & 2 \\ 4 & 6 & 1 & 3 \end{pmatrix}$$

be the exchange matrix. When defining the classification figure of sector i as

$$c_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} J(i,j), \tag{10}$$

we get the values $c_1 = 14$, $c_2 = 16$, $c_3 = 10$, and $c_4 = 11$. As $J(i,j)$ is the value of the products sold from sector i to sector j, c_i is the sum of the values of the products sold by sector i to all other sectors, such that this observable considers the problem of finding a hierarchical order from a supplier's point of view. When ordering the values of c_i according to their sizes, we get the hierarchical order $\sigma_c = (3412)$, as $\sigma_c(N)$ has to be the largest supplier. Alternatively, we can define a classification figure as

$$\tilde{c}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} J(j,i) \tag{11}$$

and thus consider this problem from a user's point of view. Here we get the values $\tilde{c}_1 = 13$, $\tilde{c}_2 = 20$, $\tilde{c}_3 = 10$, and $\tilde{c}_4 = 8$. We achieve the hierarchical order $\sigma_{\tilde{c}} = (2134)$, as $\sigma_{\tilde{c}}(1)$ has to be the largest user. The two observables c and $\tilde{c}$ consider different aspects of this problem and thus lead to different results. There is of course a way of combining these aspects by considering the differences between the entries just as in the last section and defining

$$\hat{c}_i = c_i - \tilde{c}_i = \sum_{j=1}^{N} (J(i,j) - J(j,i)). \tag{12}$$

Here we get the values $\hat{c}_1 = 1$, $\hat{c}_2 = -4$, $\hat{c}_3 = 0$, $\hat{c}_4 = 3$ and thus the order $\sigma_{\hat{c}} = (2314)$. When solving the RPB optimization problem for this toy instance, we get the unique optimum configuration $\sigma_{\mathrm{RPB}} = (2431)$ with a ground state energy value of 22. The other configurations exhibit cost function values in the range $[23; 29]$.

Summarizing, we find that these different approaches lead to different hierarchical orders and that none of these simple approaches is able to produce a result which is optimum for the optimization problem proposed by Ramsey, Pierce, and Bowman. Although these simple approaches consider different aspects of a hierarchical ordering and might also have their justifications in some economic theories, we want to stick with the complex problem Ramsey, Pierce, and Bowman had to solve in their context, which intends to find a hierarchical ordering from the supplier's and the user's point of view simultaneously.

3 Computational Results for a Benchmark Instance

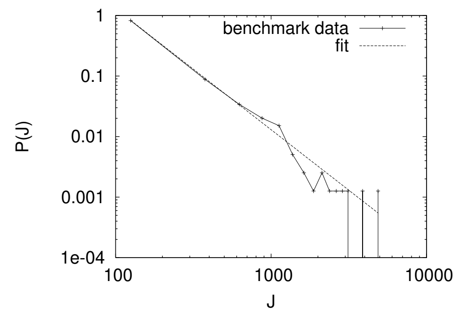

Fig. 1. The distribution function of the nondiagonal nonvanishing entries in the exchange matrix J of the benchmark instance is given as $\sim 1.3 \cdot 10^4 \times J^{-2}$.

In their publication of 1969 [1], Ramsey, Pierce, and Bowman also provided a benchmark instance comprised of a 37×37 matrix J based on the exchange of goods and services within the United States of America of the year 1947. Those entries in J that are marked as asterisks and thus classified as marginal in [1] shall be set to zero. The nonvanishing nondiagonal entries of J are power law distributed with an exponent of -2, as shown in Fig. 1. The mean value of all nondiagonal entries is ~ 108.75, the maximum is 4804. The authors were able to find an optimum solution for this benchmark instance consisting of 37 economic sectors with the ground state energy $\mathcal{H}_J(\sigma_0) = 25306$, consisting of $\mathcal{H}_K(\sigma_0) = 18650$ and $\mathcal{H}_L(\sigma_0) = 6656$.

For this publication, this benchmark instance shall be treated with the Simulated Annealing (SA) algorithm [6] and its deterministic variant [7], which is usually called Threshold Accepting (TA) [8]. When applying SA, the proposed optimization problem is considered as a classical physical system, which is gradually cooled down in an annealing process, thus being transferred from a high-energetic unordered regime to a low-energetic ordered solution. In each temperature step, several moves are applied to the system changing the configuration. When using SA, these moves are accepted according to the Metropolis acceptance criterion [9] with the acceptance probability

$$W(\sigma_{\text{current}} \to \sigma_{\text{new}}) = \begin{cases} 1 & \text{if } \Delta\mathcal{H}_J \leq 0 \\ \exp(-\Delta\mathcal{H}_J/T) & \text{otherwise} \end{cases} \tag{13}$$

with the energy difference $\Delta\mathcal{H}_J = \mathcal{H}_J(\sigma_{\text{new}}) - \mathcal{H}_J(\sigma_{\text{current}})$ between the current configuration σ_{current} and the tentative new configuration σ_{new}. T denotes the temperature. For TA, the acceptance criterion

$$W(\sigma_{\text{current}} \to \sigma_{\text{new}}) = \begin{cases} 1 & \text{if } \Delta\mathcal{H}_J \leq T \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

is used. Thus, every move is accepted which leads either to an improvement or to a deterioration of a size which must not be larger than the threshold value T. In case of rejection, one sets $\sigma_{\text{new}} = \sigma_{\text{current}}$.

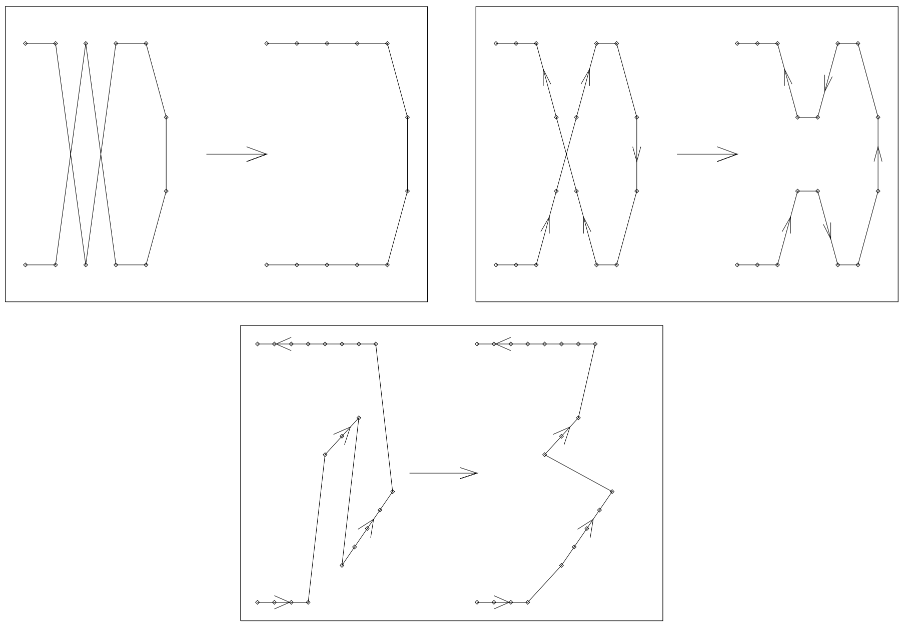

Fig. 2. Graphic illustration of the moves used for changing the configuration of the RPB model which is drawn like for a TSP with open end points: EXC (top left), L2O (top right), and L3O (bottom)

As the system size $N = 37$ of this benchmark instance is very small, it is sufficient to work with small moves only which do not change a configuration very much. Three small moves which are also commonly used for the TSP [10] have been implemented for the RPB model and are graphically illustrated in Fig. 2, in the way as if they were applied to a TSP with open end points. The various sectors of the economy are drawn as points on the plane. They are connected with edges picturing the sequence of these sectors in the current configuration. The Exchange move (EXC), which is shown in the top-left picture of Fig. 2, swaps two economic sectors in the configuration σ. The Lin-2-Opt move (L2O) [11, 12] cuts two randomly selected edges in the sequence σ, turns the partial sequence between these two cuts around, thus changing its direction, and reconnects it with the usually two other partial sequences. Please note that in contrast to the TSP, for which mostly a closed tour is considered, the RPB model has open boundary conditions, such that also the cases have to be considered that there is no partial sequence before and after the partial sequence to be turned around and the whole sequence changes its direction after the application of the L2O, or that there is only one of them. The Lin-3-Opt move (L3O) cuts three randomly selected edges in the sequence, thus usually creating four partial sequences, exchanges the positions of the second and the third partial sequence, and reconnects the partial sequences. Again, there are special cases of the L3O, in which there are overall only two partial sequences to be exchanged or three partial sequences, two of which change their positions. Each of these three move routines is called with probability $\frac{1}{3}$.

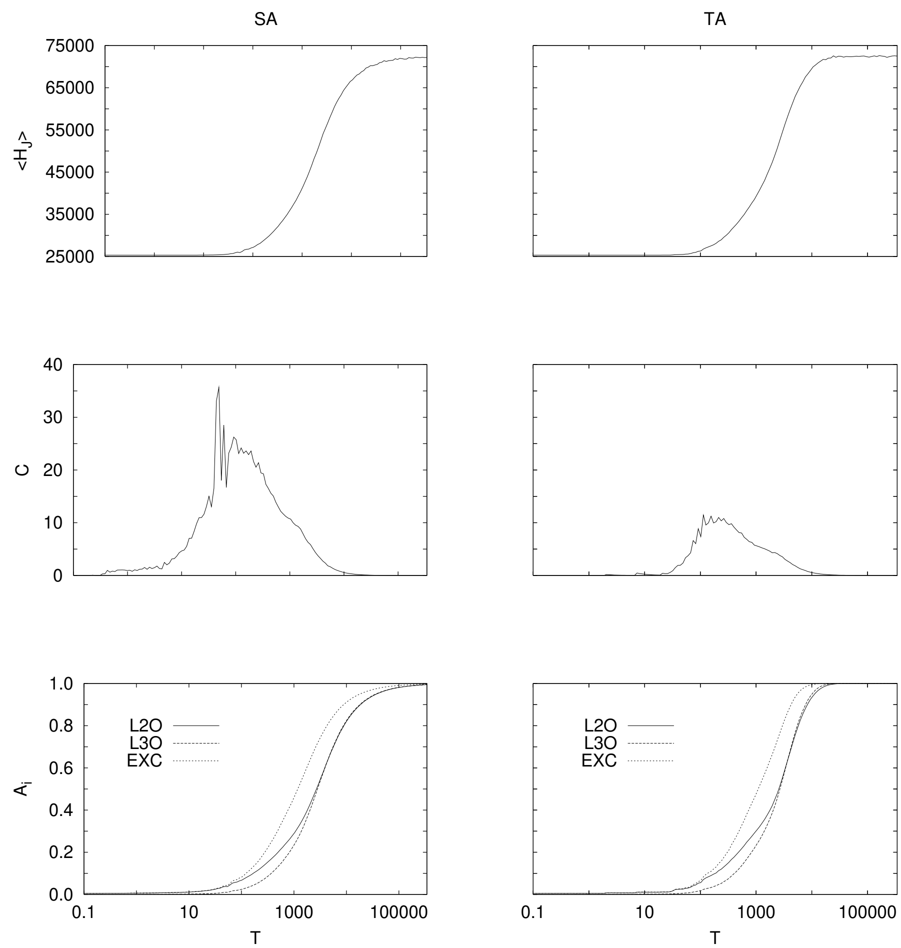

Fig. 3. Computational results for the application of SA (left) and TA (right) to the benchmark instance of the RPB problem: mean energy $\langle \mathcal{H}_J \rangle$ (top), specific heat C (middle), and acceptance rates A_i for $i \in \{$ L2O, L3O, EXC $\}$ (bottom)

Analogously to the SK model and the TSP [10], one finds that it is best to cool this benchmark instance of the RPB problem in an exponential way; a cooling factor of 0.9 is used. The initial temperature is determined automatically in a short random walk at the beginning, the final temperature is set to 5×10^{-3}. At the beginning of each temperature step, 10,000 Monte Carlo sweeps (MCS) are performed before the first measurement is taken. 5,000 measurements are taken, between which 10 MCS are performed. The results shown in Fig. 3 are averaged over these measurements. The top row of Fig. 3 shows the sigmoidal decrease of the mean energy $\langle \mathcal{H}_J \rangle$ with decreasing temperature T over three orders of magnitude of the temperature. Thus, the system orders itself on a logarithmic temperature scale. The decrease of the mean energy is steeper in the case of TA than in the case of SA. Analogous results are found for the SK model and for the TSP [10]. The middle row displays the specific heat C, which is measured via the identity $C = \mathrm{Var}(\mathcal{H}_J)/T^2$ found in thermal equilibrium. For TA, the height of the peak is much smaller than for SA, the peak lies at slightly higher values of T. For both cases, a small bulge can be seen at $T \sim 1000$, such that the question arises whether there is a small clustering and ordering effect. The bottom row shows the decrease of the acceptance rates of the various moves with decreasing T. The acceptance rate of EXC decreases slower than the acceptance rates of the L2O and of the L3O. This finding, which is in contrast to corresponding results for the TSP [10, 13], can be easily explained. When calculating the energy difference $\Delta\mathcal{H}_J$ for the decision whether to accept or reject the move, the number of addends to be summed up is of the order of the system size N for EXC, whereas it is $\mathcal{O}(N^2)$ for L2O and L3O, such that the possible values for the energy differences are much larger for L2O and L3O. (Contrarily, there are 4 addends to $\Delta\mathcal{H}_J$ when applying the L2O to symmetric TSPs, 6 addends for the L3O, and 8 addends for the EXC, independently of the system size.) For high temperatures, the acceptance rate of the L3O is slightly larger than the acceptance rate of the L2O, then the curves for the acceptance rates of the L2O and L3O cross and the decrease is steeper for the L3O than for the L2O. At low temperatures, the acceptance rate of the L2O approaches the acceptance rate of the EXC.

Nearly all optimization runs end in a configuration with the global minimum energy value 25306, but there are different configurations with this minimum value. Therefore, this benchmark instance has a degenerate ground state. The extent of such degeneracies can be studied with the parallel Searching for Backbones algorithm, which was initially developed for the TSP [14] and later on also applied to extensions of the TSP [15, 16] and to spin glass models [17]. The degeneration of the ground state of the benchmark instance studied here is restricted to the first five entries of the permutation vector σ and thus lies at the side of the largest users, whereas there is a strictly given optimum order for the largest suppliers. The largest users are non-profit organizations, amusements, scrap & miscellaneous industries, eating & drinking places, and ocean transportation, whereas the largest suppliers are transport via railroad and trucks, electric power plants, and banks and insurance companies. Please note that the results for the degeneracy and for the otherwise hierarchical order of the various economic sectors apply to this benchmark instance only. Mostly, the ground states of instances of the RPB problem will not be degenerate.

4 Conclusion and Outlook

In this paper, the properties of the widely unknown Ramsey-Pierce-Bowman problem are investigated: they introduced a model for finding a hierarchical order of the various sectors of an economy, in the viewpoint of users, which need to buy a lot of products from other sectors for their own production, and of suppliers, which earn money by providing their goods as preproducts to other sectors. According to different economic theories, either the largest suppliers or the largest users are the most important sectors of the economy. Therefore, it is of great necessity to know them. It turns out that this problem is located between the Traveling Salesman Problem, in which also an optimum sequence has to be found, and infinite-dimensional spin glass models, which make also use of the complete upper triangular part of the interaction matrix. After the derivation of some interesting properties of this model and showing that it cannot be trivially solved, the original benchmark instance is optimized using Simulated Annealing and the related Threshold Accepting optimization technique. It is found that the solution of this problem provided in [1] is indeed optimal, but that the benchmark instance exhibits a degenerate ground state. The computational results are similar to those obtained for the TSP and the SK model.

We will continue the investigation of this model, especially by comparing these results for data of the year 1947 with results for more recent years in order to see the change of the importance of the various sectors for the US economy. We expect that e.g. the scrap industry has changed its role from a user to an important supplier. The RPB model has also many other applications: in the foreign trade, suppliers have a trade deficit, whereas the users have a trade surplus. In political sciences, the movement of voters between the various parties is studied, here the suppliers are those parties which lose votes to other parties. A further application is the investigation of migration processes of peoples: users are immigration countries, suppliers emigration countries. Furthermore, we want to investigate the properties of the exchange matrices of these and related problems and see whether they also exhibit scale free properties like the exchange matrix of the benchmark instance we investigated here. Finally, we also want to study the problem of detecting the importance of an economic sector as user or supplier also with other algorithms, e.g. with a flow analysis using the infomap algorithm [19].

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