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Weighted variance swaps hedge against impermanent loss

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Impermanent Loss in Decentralized Finance can be hedged with weighted variance swaps

1. Introduction

Decentralized Exchanges (DEXes) allow users to trade in a fully noncustodial manner. Traders can directly swap their digital currencies using a smart contract, a program running on the blockchain, rather than trusting a central counterparty with their funds. In the early stages, the low throughput of blockchains required another trading model than the traditional order book approach, which gave rise to Automated Market Makers (AMMs). An AMM is a smart contract that determines the price for which traders can swap their digital currency against another digital currency. For the trade to happen, liquidity providers lock digital currencies into a smart contract, the liquidity pool. The AMM deposits the trader's digital currency into the liquidity pool and pays the trader with the other digital currency from the liquidity pool according to the price provided by the AMM. This alters the amounts owned by liquidity providers. In turn, liquidity providers earn trading fees, cf. Mohan (2022). In a *Constant Function Market*, the AMM determines the price via a so-called trading function – a function of the liquidity pool's reserves – so that the value of the trading function given the post-trade reserves equals its value given the pre-trade reserves.

Typically, liquidity provision is segregated into pairs of tokens that can be swapped, e.g. there is one pool for BTC-USD and another one for ETH-USD for a given AMM. Hence, we will focus on two assets in this article. Most AMMs require that liquidity providers deposit the pair subject to equal value (e.g. when BTC is at 20 000 USD, the ratio of BTC to USD deposited has to be 1:20 000), so that liquidity providers cannot choose the relative amount of the digital currency they deposit.

When the exchange rate of the digital currencies moves, the liquidity provider's portfolio is subject to price risk. In fact, neglecting trading fees, liquidity providers are worse off when the price moves away from the original exchange rate, compared to the buy-and-hold investor. This loss relative to the buy-and-hold portfolio is termed *Impermanent Loss*. Some DEXes have tried to address this issue by changing their AMM design using one of the following two approaches. One approach is to adjust the AMM pricing formula, see, e.g. Balancer v2 (2022), which in turn has negative consequences on slippage. Another approach is to reward liquidity providers with the protocol token in the hope of mitigating the losses incurred from liquidity provision, see, e.g. Bancor v3 (2022), which works out for the liquidity provider as long as the protocol's token is valuable enough. To the best of our knowledge, there is no DEX offering a hedge against Impermanent Loss. However, with the rise of derivatives in DeFi, hedging Impermanent Loss has become possible.

The success of AMMs is often measured by the value of the funds locked into the contracts for liquidity provision, termed Total Value Locked (TVL). At the time of this writing, TVL in DEXes stood at USD 41bn, while their combined trading volume in December 2022 was USD 45bn.†

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† As retrieved from Defillama on 9 January, 2023

Currently, many platforms charge fixed proportional transaction fees that are paid to liquidity providers. We employ arguments from risk-neutral valuation to determine the fair price of providing liquidity in a complete market.

1.1. Related literature

Early analyses of Constant Product Markets can be found in Angeris *et al.* (2020) and Angeris *et al.* (2021c). Evans (2020) discusses the returns of liquidity providers in geometric mean market makers, which generalize the concept of Constant Product Markets to pools containing several tokens with dynamic weights. Mohan (2022) provides a comprehensive and systematic overview of Automated Market Makers, and Lipton and Sepp (2021) present an AMM cross-settlement mechanism for Central Bank Digital Currencies (CBDC). Clark (2020) derives the replicating portfolio of a Constant Product Market and describes a static hedge for the dollar-value of liquidity provision. While KPMG China (2021) claim that Impermanent Loss (synonymously, Divergence Loss) can be hedged with a long straddle, this does not apply in general. A general overview of Decentralized Finance can be found in Schär (2021) and Lipton and Treccani (2021).

There is a rich academic literature on modelling volatility, cf. Gatheral (2012), Bossu (2004) and the references therein for a general overview, and Fukasawa (2014) for an exposition on (weighted) variance swaps. Connections of Constant Function Markets with variance swaps were described in Angeris *et al.* (2021a, 2021b).

1.2. Outline

We start our analysis with considerations on the risk-neutral valuation of liquidity provision fees in Constant Product Markets in Section 2. The close connection between (the hedging of) Impermanent Loss and variance swaps and gamma swaps is demonstrated in Section 4. Furthermore, we derive alternative trading functions that lead to market structures in which Impermanent Loss can be hedged with a variance swap or a gamma swap, respectively, see Proposition 6.4. We finally show in Proposition 6.6 that any concave payoff can be replicated in a Constant Function Market with a suitably defined trading function.

Throughout the text, we give practical examples that illustrate the relevance and consistency of our results with empirical observations.

1.3. Main contributions

This paper contributes to the existing literature by demonstrating that Impermanent Loss, a key concept for Decentralized Exchanges, can be hedged with a weighted variance swap. This allows us to put Impermanent Loss into context with variance swaps and gamma swaps, which are central objects of research in volatility modeling and trading. Moreover, we define a one-parameter family of Constant Function Markets whose Impermanent Loss can be hedged with weighted variance swaps.

2. Liquidity provision

We analyze Constant Function Markets, often also called Constant Function Market Makers (CFMMs). Examples of Constant Function Markets implementations are Uniswap v2 or Balancer, see Martinelli and Mushegian (2019).

Definition 2.1 Constant Function Markets and Trading Function *Constant Function Markets are a type of automated market maker defined by its reserves and its trading function. The reserves are the tokens available to the smart contract given by* $x_0 \in \mathbb{R}_+$ *of the token to be traded and* $y_0 \in \mathbb{R}_+$ *of the numéraire token. The trading function maps the pair of reserves* $(x, y) \in \mathbb{R}^2_+$ *and a trade* $(\Delta x, \Delta y) \in \mathbb{R}^2$ *to a scalar:* $\ell : \mathbb{R}^2_+ \times \mathbb{R}^2 \to \mathbb{R}$. *An admissible CFMM trade* $(\Delta x, \Delta y)$ *is implicitly defined by*

$$\ell((x, y), (\Delta x, \Delta y)) \equiv \ell((x, y), (0, 0)), \tag{1}$$

cf. Angeris et al. (2020). That is, if the trader sends Δx *tokens to the CFMM smart contract, she will receive the amount* Δy *of the other token such that Equation 1 is satisfied.*

Constant Product Markets (cf. Angeris and Chitra 2020, Mohan 2022) are a particular form of Constant Function Markets.

Definition 2.2 Constant Product Markets *A Constant Product Market is a Constant Function Market whose trading function is defined as*

$$\ell((x, y), (\Delta x, \Delta y)) := (x + \Delta x)(y + \Delta y). \tag{2}$$

If the Constant Product Market charges traders transaction fees, then its trading function takes the form

$$\ell((x, y), (\Delta x, \Delta y)) := (x + (1 - \tau^X)\Delta x)(y + (1 - \tau^Y)\Delta y), \tag{3}$$

where $\tau^X = \tau \mathbf{1}_{\{\Delta x > 0\}}$, $\tau^Y = \tau \mathbf{1}_{\{\Delta y > 0\}}$, *and* $\tau \in [0, 1)$ *is a constant representing transaction fees.*

Depending on the implementation of the smart contract for a particular Constant Product Market, fees can be applied to the incoming token or the outgoing token, cf. Evans *et al.* (2021). The subsequent analysis will focus on Constant Function Markets without fees; we will return to the subject of transaction fees in Section 5.

Definition 2.3 Exchange Rate *We define the exchange rate in a Constant Function Market as the no-fee infinitesimal price*

$$\lim_{\Delta x \to 0} \frac{|\Delta y|}{|\Delta x|}. \tag{4}$$

In a Constant Product Market, where $(x_0 + \Delta x)(y_0 + \Delta y) \equiv x_0 y_0$, the infinitesimal exchange rate is given as $S_0 := y_0/x_0$.

2.1. Constant product markets

For Constant Product Markets, we define the constant

$$L := \sqrt{x_0\, y_0}, \tag{5}$$

where $x_0 > 0$, $y_0 > 0$ denote the initial amounts of token reserves of the traded tokens in the liquidity pool (LP). Per definition of a Constant Product Market, L must remain constant throughout, i.e. any transaction $(\Delta x, \Delta y)$ performed by a liquidity taker occurs according to $(x_0 + \Delta x)\,(y_0 + \Delta y) = L^2$. The LP's initial exchange rate between asset X and asset Y being $S_0 = y_0/x_0$, we notice that

$$x_0 = \frac{L}{\sqrt{S_0}}, \quad y_0 = L\sqrt{S_0}. \tag{6}$$

We assume that there is only one participant in the LP, and that there are no changes in the amount of liquidity (i.e. there are no deposits or withdrawals) until the terminal time T. Then, denoting by x_T and y_T the amounts of token reserves at T, we have $x_T y_T = L^2$. The exchange rate at T is $S_T = y_T/x_T$. Now we can write the value of the liquidity provider's portfolio in the absence of fees and other liquidity providers as

$$V_{LP}(0) = y_0 + x_0\, S_0 = 2L\sqrt{S_0} \implies V_{LP}(T) = 2L\sqrt{S_T}. \tag{7}$$

2.2. *Multiple liquidity providers*

So far, we have restricted the setting to a single liquidity provider. We now show that the value of one liquidity provider's claim to the pool is independent of other liquidity providers' actions (i.e. injections and withdrawals).

In practice, a liquidity provider deposits equivalent amounts of tokens to the pool and receives 'pool share tokens', so that the amount of pool shares relative to the total amount of issued pool shares corresponds to the amount deposited relative to the total pool value. By redeeming her pool share tokens, the liquidity provider will recover a commensurate amount of tokens from the liquidity pool.

PROPOSITION 2.1 *The value of a liquidity provider's pool share is not affected by the entry and exit of other liquidity providers.*

Proof Initially, let us assume that there are two liquidity providers, A and B. Liquidity provider A contributes her tokens at time $t = 0$, while the second liquidity provider, B, does so at time $t_B \in (0, T)$.

Let $L := \sqrt{(x_A + x_B)(y_A + y_B)}$ denote the liquidity invariant of the pool combined at time $t = t_B$. In particular, x_B, y_B are the reserves deposited by liquidity provider B at the time of injection, and x_A, y_A are the reserves of the liquidity pool established earlier by liquidity provider A. Since injections/withdrawals need to correspond with the pool's exchange rate, we have

$$\frac{y_B}{x_B} = S_{t_B} = \frac{y_A}{x_A},$$

which ensures the continuity of the exchange rate

$$\frac{y_A + y_B}{x_A + x_B} = S_{t_B} = \frac{y_B}{x_B} = \frac{y_A}{x_A}. \tag{8}$$

From (7), we know that the combined pool's wealth at time T equals

$$V_{LP}(T) = 2L\sqrt{S_T}.$$

On the other hand, the liquidity pool's share of liquidity provider B is given by

$$w_B := \frac{y_B}{y_A + y_B}.$$

Therefore, the value of liquidity provider B's share in the combined liquidity pool at time T is

$$\begin{aligned} V_{LP,B}(T) &= w_B V_{LP}(T) \\ &= \frac{y_B}{y_A + y_B} 2\sqrt{(x_A + x_B)(y_A + y_B)}\sqrt{S_T} \\ &= 2\frac{y_B}{\sqrt{y_A + y_B}}\sqrt{x_A + x_B}\sqrt{S_T} \\ &= 2y_B\sqrt{\frac{x_B}{y_B}}\sqrt{S_T} \\ &= 2\sqrt{x_B y_B}\sqrt{S_T}, \end{aligned}$$

the penultimate equality holding by (8). The last expression is precisely the value of an individual liquidity pool established separately by B. Similarly, one can show that the value of A's share in the combined liquidity pool is not affected by B's contribution. This argument can be generalized by induction, which proves the proposition. ■

3. Risk-neutral valuation of liquidity provision

3.1. *Binomial model with one period*

Let us consider a one-period model with two times, $t \in \{0, T\}$. At time T, S can take either of the following two values: $S_T = S_0\, u$ or $S_T = S_0\, d = S_0/u$ $(d := 1/u)$. We define the random variable $K_T := S_T/S_0$. Furthermore, let the risk-free rate be equal to zero: $r = 0$. Let $V_{LP}(.)$ denote the wealth of the liquidity provider in the LP, and let $V_{BH}(.)$ be the corresponding buy-and-hold strategy (the wealth of a HODLer†).

The value of the liquidity provider portfolio is given by (7), while

$$V_{BH}(T) = y_0 + x_0\, S_T = L\sqrt{S_0} + x_0\, S_T = L\sqrt{S_0}\,(1 + K_T).$$

Next, we show how to construct a portfolio that consists of liquidity provision $V_{LP}(.)$ and a straddle G, which replicates the buy-and-hold strategy $V_{BH}(.)$. For this purpose, let $V_\Delta(t) := V_{LP}(t) + \Delta G_t$. We seek to determine $\Delta \in \mathbb{R}$ such that

$$V_\Delta(T) = V_{BH}(T),$$

independently from the realization of $S(.)$ at terminal time T.

Let us derive the value of Δ:

$$\begin{aligned} &V_\Delta(T) = V_{LP}(T) + \Delta G_T = V_{BH}(T) \\ &\implies 2L\sqrt{S_T} + \Delta|S_T - S_0| = y_0\,(1 + K_T) \end{aligned}$$

† The term 'HODLer', in the jargon of the crypto community, describes a crypto investor following a buy-and-hold strategy. 'HODLing' is the corresponding activity.

$$\implies \Delta = \frac{y_0(1-2\sqrt{K_T}+K_T)}{S_0\,|K_T-1|} = \frac{L}{\sqrt{S_0}}\frac{|1-\sqrt{K_T}|^2}{|1+\sqrt{K_T}|\,|1-\sqrt{K_T}|}$$

$$\implies \Delta = \frac{|1-\sqrt{K_T}|}{1+\sqrt{K_T}}\,x_0.$$

It is straightforward to see that the value of Δ does not depend on whether there is an upward ($K_T = u$) or a downward jump ($K_T = 1/u$).

REMARK 3.1 Bearing in mind that the liquidity provider and the HODLer start with the same wealth, i.e. $V_{LP}(0) = V_{BH}(0) = 2L\sqrt{S_0}$, it is evident that the wealth of the former, no matter the future state of the world, will fall short of the wealth of the latter. This phenomenon is known as *Impermanent Loss* (or *Divergence Loss*), usually defined as the relative performance difference between liquidity provision and HODLing, i.e.

$$\hat{IL} = \frac{V_{LP}(T) - V_{BH}(T)}{V_{BH}(T)}.$$

For our purposes, however, we specify Impermanent Loss as

$$IL := IL(T) = V_{BH}(T) - V_{LP}(T). \quad (9)$$

3.1.1. Numerical example. We assume that $y_0 = x_0 = 100$, so that $S_0 = 1$. Moreover, $u = 5/4$. At time $t = T$, there are two possibilities: $S_T = S_u = 5/4$ or $S_T = S_d = 4/5$. The risk-neutral probabilities are calculated as $\pi_u = 4/9, \pi_d = 5/9$. Consequently, the straddle premium, being the sum of the call premium and the put premium, equals $G_0 = 2/9$.

Then $V_{LP}(0) = 2L\sqrt{S_0} = 200$ and

$$V_{LP}(T) = 2L\sqrt{S_0\,K_T} \approx \begin{cases} 223.6068 & \text{if } K_T = u \\ 178.8854 & \text{if } K_T = 1/u. \end{cases}$$

The HODLer's wealth equals the LP's initially; however,

$$V_{BH}(T) = L\sqrt{S_0}\,(1+K_T) = \begin{cases} 225 & \text{if } K_T = u \\ 180 & \text{if } K_T = 1/u. \end{cases}$$

The formula for Δ yields $\Delta \approx 0.0557$, so that the initial value of the hedged portfolio $V_\Delta(.)$ is given by

$$V_\Delta(0) = V_{LP}(0) + \Delta\,G_0 \approx 201.2384.$$

In other words, a rational participant in this LP would expect to receive no less than $\Delta\,G_0 \approx 1.2384$ in fees for providing liquidity during this period. Otherwise, she would be better off pursuing a buy-and-hold strategy.

3.2. *Continuous model*

Let the exchange rate $S_t = y_t/x_t$ be a diffusion process. We define $K_t := S_t/S_0$, for $t \in [0, T]$. In the continuous-time limit, the wealth of the HODler and the liquidity provider equals, respectively,

$$V_{BH}(t) = y_0\,(1+K_t) \quad (10)$$

$$V_{LP}(t) = 2y_0\sqrt{K_t}. \quad (11)$$

We consider a liquidity provider who wishes to hedge against Impermanent Loss at a finite maturity $T < \infty$. Her goal is to compensate the shortfall of providing liquidity versus HODLing using a European-style contingent claim whose terminal payoff is

$$H_T := V_{BH}(T) - V_{LP}(T) = y_0\left(1 + K_T - 2\sqrt{K_T}\right)$$
$$= y_0\left(\sqrt{K_T} - 1\right)^2.$$

Let $(\Omega, \mathcal{F} = (\mathcal{F}_t)_t, \mathbb{P})$ be a complete probability space. In a complete market, the fair price of a contingent claim making up for Impermanent Loss (9) is given by the present value of the corresponding conditional expected payoff, i.e. for $t < T$,

$$H(t) = y_t\,\mathbb{E}\left[\left(\sqrt{K_T} - 1\right)^2 \mid \mathcal{F}_t\right]$$
$$= x_t\,\mathbb{E}\left[\left(\sqrt{S_T} - \sqrt{S_t}\right)^2 \mid \mathcal{F}_t\right] \quad (12)$$

where the expectation is taken under the risk-neutral measure.† Since investing in the powered power straddle $H(.)$ guarantees that the Impermanent Loss vanishes at time $t = T$, $H(t)$ is a lower bound for the cumulative transaction fees that a rational liquidity provider should have earned by time t.

REMARK 3.2 Note that the payoff of this contingent claim differs from that of a European ATM straddle: indeed, it is the sum of an ATM powered power call option with payoff

$$\widetilde{C}_T = x_0\,\max\left(\sqrt{S_T} - \sqrt{S_0}, 0\right)^2$$

and an ATM powered power put option with payoff

$$\widetilde{P}_T = x_0\,\max\left(\sqrt{S_0} - \sqrt{S_T}, 0\right)^2.$$

3.2.1. Numerical example (continued). Using the volatility inferred from the magnitude of an upward jump ($u = 5/4$) in our numerical example, $u = e^{\sigma/\sqrt{\Delta t}}$, we get that $\sigma \approx 0.2236$. A Monte-Carlo simulation applied to estimate (12) with 1 million paths, $r = 0$, $T = 1$, $S_0 = 1$ yields the premium $H_0 \approx 1.2414$. The premium calculated with the binomial model, $\Delta G_0 \approx 1.2384$, is not too far off this mark.

3.3. *Dynamic hedging in a Black-Scholes market*

Assume that the external market‡ is liquid and that the price S_t follows the geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t, \quad S_0 > 0$$

† Recall that we assume the risk-free rate to be zero.
‡ We refer to the collection of all trading venues (except the LP under consideration) where the two tokens X, Y can be swapped as the *external market*.

for $\mu, \sigma \in \mathbb{R}$, $\sigma > 0$, where $W = (W_t)_t$ denotes a one-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that the bivariate function

$$p(s,t) = s^\alpha \exp\left\{-\frac{1}{2}\alpha(1-\alpha)\sigma^2(T-t)\right\}$$

solves the Black-Scholes PDE

$$\frac{\partial p}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 p}{\partial s^2} = 0, \quad p(s,T) = s^\alpha$$

for any $\alpha \in (0,1)$, cf. Fukasawa (2014). In particular, for $\alpha = 1/2$,

$$\begin{aligned}\sqrt{S_T} = p(S_T, T) &= p(S_0, 0) + \int_0^T \frac{\partial p}{\partial s}(S_t, t)\,\mathrm{d}S_t \\ &= \sqrt{S_0}\, e^{-\sigma^2 T/8} + \int_0^T \frac{\partial p}{\partial s}(S_t, t)\,\mathrm{d}S_t.\end{aligned}$$

Therefore,

$$\begin{aligned}x_0\left(\sqrt{S_T} - \sqrt{S_0}\right)^2 &= x_0\left(S_T + S_0 - 2\sqrt{S_T S_0}\right) \\ &= x_0 S_T + y_0 - 2y_0 e^{-\sigma^2 T/8} \\ &\quad - 2L\int_0^T \frac{\partial p}{\partial s}(S_t, t)\,\mathrm{d}S_t.\end{aligned}$$

Since $y_0 = x_0 S_0$, this is equivalent to

$$\begin{aligned}x_0\left(\sqrt{S_T} - \sqrt{S_0}\right)^2 &= 2y_0\left(1 - e^{-\sigma^2 T/8}\right) + x_0\,(S_T - S_0) \\ &\quad - 2L\int_0^T \frac{\partial p}{\partial s}(S_t, t)\,\mathrm{d}S_t. \qquad (13)\end{aligned}$$

We conclude by noting that, taking expectations under the risk-neutral measure,

$$x_0\,\mathbb{E}\left[\left(\sqrt{S_T} - \sqrt{S_0}\right)^2\right] = 2y_0\left(1 - e^{-\sigma^2 T/8}\right). \qquad (14)$$

This holds because the second and the third term on the right-hand side of (13) vanish, the latter due to its being a stochastic integral with respect to a square-integrable martingale.

The amount $2y_0(1 - e^{-\sigma^2 T/8}) \approx y_0\sigma^2 T/4$ can serve as a benchmark for the fair fee for providing liquidity until maturity T even if the external market is illiquid (or does not even exist).†

3.3.1. Numerical example (continued). Recall from our previous example that $x_0 = y_0 = 100$, $T = 1$, and $\sigma = \log(u) = \log(5/4)$. Using (13), we find that

$$2y_0\left(1 - e^{-\sigma^2 T/8}\right) \approx 1.2410,$$

whereas

$$y_0\frac{\sigma^2 T}{4} \approx 1.2448.$$

Both figures are very close to the ones found previously with a Monte-Carlo simulation, and for the binomial model.

† A similar formula was found in Angeris *et al.* (2021c).

4. Static hedging with weighted variance swaps

4.1. Model-free hedging of impermanent loss

Following Bossu (2004, Problem 3.2), any contingent claim with a payoff function $f(S_T)$ that is twice continuously differentiable can be perfectly hedged with (a continuum of) European calls and puts. The premium of such a contingent claim is given by

$$f_0 = f(S_0) + \int_0^{S_0} f''(k)\,p_0(k)\,\mathrm{d}k + \int_{S_0}^\infty f''(k)\,c_0(k)\,\mathrm{d}k,$$

where $p_0(k)$, $c_0(k)$ denote puts and calls struck at a continuum of strikes $k > 0$. Due to (12), the payoff reads

$$\begin{aligned}f(k) &:= x_0\left(\sqrt{k} - \sqrt{S_0}\right)^2 \\ \Longrightarrow f'(k) &= 2\left(\sqrt{k} - \sqrt{S_0}\right)\frac{x_0}{2\sqrt{k}} = x_0\left(1 - \sqrt{\frac{S_0}{k}}\right) \\ \Longrightarrow f''(k) &= \frac{x_0}{2}\sqrt{\frac{S_0}{k^3}}.\end{aligned}$$

It follows that Impermanent Loss (9) can be hedged using vanilla puts and calls:

$$H_0 = \frac{L}{2}\left[\int_0^{S_0} \frac{p_0(k)}{k^{3/2}}\,\mathrm{d}k + \int_{S_0}^\infty \frac{c_0(k)}{k^{3/2}}\,\mathrm{d}k\right]. \qquad (15)$$

REMARK 4.1 In particular, (15) implies that Impermanent Loss can be hedged statically and in a *model-free* manner purchasing European puts and calls in pre-defined quantities, cf. Appendix 2 for a computation using actual options data. In Clark (2020, Section 4), the author derived the replicating portfolio for the liquidity provider (V_{LP}, in our notation).

Let us write, in the spirit of Fukasawa (2014),

$$\int_0^T g(S_u)\,\mathrm{d}\langle\log S\rangle_u = f_g(S_T) - f_g(S_0) - \int_0^T f_g'(S_u)\,\mathrm{d}S_u, \qquad (16)$$

which is a consequence of the Itô-Tanaka-Formula.‡ Here, g is a locally integrable function, and f_g is a function satisfying

$$f_g(x) = 2\int_1^x \int_1^y \frac{g(z)}{z^2}\,\mathrm{d}z.$$

There are two well-known special cases (cf. Fukasawa 2014 and the references therein).

First, if $g(z) \equiv 1$, we recover the variance swap, which is well-known to be hedged statically with a log contract (cf. Neuberger 1994, Demeterfi *et al.* 1999, Gatheral 2012):

$$\mathbb{E}[\langle X\rangle_T] = -2\,\mathbb{E}\left[\log\left(\frac{S_T}{S_0}\right)\right],$$

where $X_t := \log K_t$ for all $t \in [0, T]$.

‡ We use the symbols $\langle . \rangle$ to denote quadratic variation.

The gamma swap, on the other hand, arises if $g(z) = z$, so that

$$\frac{1}{S_0}\mathbb{E}\left[\int_0^T S_t \, \mathrm{d}\langle \log(S)\rangle_t\right] = 2\,\mathbb{E}\left[\frac{S_T}{S_0}\log\frac{S_T}{S_0}\right],$$

the right-hand side of which is also known as 'entropy contract'.

Now, if we consider $g(z) = \sqrt{z}$, which yields $f_{\sqrt{\cdot}}(x) = 4(\sqrt{x}-1)^2$, then we see that Impermanent Loss can be hedged using a weighted variance swap, cf. Fukasawa (2014):

$$\mathbb{E}\left[\int_0^T \sqrt{S_t}\, \mathrm{d}\langle \log(S)\rangle_t\right] = \frac{4}{\sqrt{S_0}}\,\mathbb{E}\left[\left(\sqrt{S_T}-\sqrt{S_0}\right)^2\right].$$

If we define the parameterized family of functions

$$g_\alpha(z) = z^\alpha, \quad \alpha \in \mathbb{R}, \tag{17}$$

then we can consider the Impermanent Loss hedge ($\alpha = 1/2$) to lie between a variance swap ($\alpha = 0$) and a gamma swap ($\alpha = 1$).

Proposition 4.2 *In a Constant Product Market, Impermanent Loss (9) can be hedged statically with a weighted variance swap with exponent $\alpha = 1/2$.*

Remark 4.3 It is intriguing that weighted variance swaps in fact have explicit representations in the Heston model, cf. Appendix 1.

4.2. *Approximate hedging with variance and gamma swaps*

Market making in traditional order-book-based markets involves actively participating in the market, e.g. by adjusting orders after an event according to the market maker's strategy. By contrast, liquidity provision in DeFi is envisioned to be a passive way to enable trading. In this spirit, it is appropriate to look for a static hedge that the liquidity provider can purchase when starting their investment.

In terms of (17), the Impermanent Loss hedge lies between the gamma swap and the variance swap. Therefore, we proceed by searching for an approximation of the Impermanent Loss hedge via gamma swap, and then via variance swap. We find that the two approximations sandwich the Impermanent Loss hedge and we suggest a convex combination of the two approximations to better approximate the Impermanent Loss hedge.

4.3. *Approximate hedge with gamma swaps*

Using Equation (15) we write H_T as

$$H_T = \frac{x_0\sqrt{S_0}}{2}\left[\int_0^{S_0}\frac{p_T(k)}{k^{3/2}}\,\mathrm{d}k + \int_{S_0}^{\infty}\frac{c_T(k)}{k^{3/2}}\,\mathrm{d}k\right].$$

This is equivalent to

$$H_T = \frac{x_0\sqrt{S_0}}{2}\left[\int_{S_0}^{S_T}\frac{S_T-k}{k^{3/2}}\,\mathrm{d}k\right]. \tag{18}$$

We move $\sqrt{S_0}$ into the integral and replace it with $\sqrt{k}$ to get

$$H_T^{(\Gamma)} = \frac{x_0}{2}\left[\int_{S_0}^{S_T}\frac{S_T-k}{k}\,\mathrm{d}k\right]. \tag{19}$$

From this expression, we see that when S_T is close to S_0, $H_T^{(\Gamma)}$ is close to H_T, and

$$H_T^{(\Gamma)} > H_T \text{if } S_T > S_0, \quad \text{while } H_T^{(\Gamma)} < H_T \text{if } S_T < S_0,$$

meaning that if we used $H^{(\Gamma)}$ in our hedging approach, we would over-hedge if the terminal prize S_T ends up above the beginning of period spot, and we would under-hedge if $S_T < S_0$. Integration of (19) yields

$$\begin{aligned} H_T^{(\Gamma)} &= \frac{x_0}{2}\int_{S_0}^{S_T}\frac{S_T}{k}\,\mathrm{d}k - \frac{x_0}{2}\int_{S_0}^{S_T}\mathrm{d}k = \frac{x_0}{2}\left[S_T\log(k) - k\right]\Big|_{k=S_0}^{k=S_T} \\ &= \frac{x_0}{2}\left[S_T\log\frac{S_T}{S_0} - (S_T - S_0)\right]. \end{aligned} \tag{20}$$

The second term of Equation (20) is equal to the payoff at the expiration of half a forward contract on the underlying S with delivery price S_0, which is linear in S_T. The forward contract can be valued without any knowledge about the volatility of S. On the other hand, the first term of Equation (20) corresponds to the payoff of an entropy contract.

From Fukasawa (2014), we know that the value of a gamma swap is given by the expectation of the entropy contract

$$\mathcal{E}_0 = 2\,\mathbb{E}\left[\frac{S_T}{S_0}\ln\frac{S_T}{S_0}\right].$$

Plugging this into (20) gives

$$H_0^{(\Gamma)} = \frac{x_0}{4}\left[S_0\,\mathcal{E}_0 - 2\,F_0\right],$$

where F_0 corresponds to a forward contract with payoff $(S_T - S_0)$. An imperfect hedge for Impermanent Loss could thus be to purchase a gamma swap, yielding an over-hedge if $S_T > S_0$, and an under-hedge otherwise.

4.4. *Approximate hedge with variance swaps*

Again we start with H_T given by Equation (18) and construct an approximation that is close to H_T when S_T is close to S_0. To do so, we write

$$H_T = \frac{x_0 S_0}{2}\int_{S_0}^{S_T}\frac{1}{k^{3/2}\sqrt{S_0}}(S_T - k)\,\mathrm{d}k$$

and substitute $\sqrt{S_0}$ in the denominator inside the integral with $\sqrt{k}$ to obtain an approximation

$$H_T^{(\upsilon)} = \frac{x_0 S_0}{2}\int_{S_0}^{S_T}\frac{S_T-k}{k^2}\,\mathrm{d}k. \tag{21}$$

We notice that H_T is close to $H_T^{(\upsilon)}$ when S_T is close to S_0, and

$$H_T^{(\upsilon)} < H_T \text{ if } S_T > S_0, \quad \text{while } H_T^{(\upsilon)} > H_T \quad \text{if } S_T < S_0.$$

Therefore, if we used $H^{(\upsilon)}$ in our hedging approach, we would under-hedge if the terminal prize S_T ended up above the initial spot, and we would over-hedge otherwise. Integrating (21) yields

$$H_T^{(\upsilon)} = \frac{x_0}{2}\left[-S_0 \log\frac{S_T}{S_0} + (S_T - S_0)\right]. \qquad (22)$$

Again, the second term corresponds to a forward contract – in fact, it is the same contract as for the gamma-contract hedge $H_T^{(\Gamma)}$, only with the opposite sign. The first part corresponds to the payoff of a variance swap. The value of a variance swap (see Bossu (2004, Section 5-2)) is given by the expectation

$$\Upsilon_0 = -2\,\mathbb{E}\left[\log\frac{S_T}{S_0}\right].$$

The first term on the right-hand side of Equation (22) thus corresponds to S_0 quarters of variance swaps and hence

$$H_0^{(\upsilon)} = \frac{x_0}{4}\left[S_0\Upsilon_0 + 2\,(S_T - S_0)\right],$$

4.5. *Approximate hedge with a weighted average*

Since $H_T^{(\upsilon)}$ over-hedges whenever $H_T^{(\Gamma)}$ under-hedges and vice-versa (cf. Figure 1), we do a convex combination of a Gamma swap and a variance swap to approximate the perfect hedge H_0:

$$\begin{aligned} H_0^{(\vartheta)} &= wH_0^{(\upsilon)} + (1-w)H_0^{(\Gamma)} \\ &= \frac{y_0}{4}\left[w\Upsilon_0 + (1-w)\mathcal{E}_0\right] + \frac{x_0}{2}F_0, \end{aligned}$$

where $w \in [0, 1]$ and F_0 correspond to a forward contract with payoff $(S_T - S_0)$. For each terminal value S_T, we can find a portfolio weight w so that the value of our hedge portfolio $H^{(\vartheta)}$ at maturity equals that of the perfect hedge; that is, $H_T^{(\vartheta)}(w, S_T) = H_T(S_T)$, see Figure 1. Building on this, we can find a w^* that minimizes the mean-squared hedging error for a pre-specified set of terminal values $s \in \mathcal{S}$:

$$w^* = \arg\min_w \sum_{s\in\mathcal{S}} \left(H_T^{(\vartheta)}(w,s) - H_T(s)\right)^2, \qquad (23)$$

where we write H_T as a function of S_T.

4.5.1. Example.

We set $S_0 = 100$ and define $\mathcal{S} = \{s \in \mathbb{N} \mid s \geq 10, s \leq 190\}$. This leads to $w^* = 0.61$. Figure 2 plots the payoff of the perfect hedge and its approximation using w^*.

5. Hedging impermanent loss in the presence of fees

If, in Definition 2.2, we assume that there are non-zero fees, i.e. $\tau > 0$, then we can show that

$$\Delta y = -\frac{1-\tau^X}{1-\tau^Y}\, S\Delta x \frac{x}{x + (1-\tau^X)\Delta x},$$

so that the exchange rate is computed as

$$\lim_{\Delta x\downarrow 0}\frac{|\Delta y|}{|\Delta x|} = (1-\tau)S, \quad \lim_{\Delta x\uparrow 0}\frac{|\Delta y|}{|\Delta x|} = \frac{1}{1-\tau}S, \qquad (24)$$

the first expression denoting the bid price (for selling Δx to the pool in exchange for Δy), and the second the corresponding ask price.

In the presence of fees, the liquidity pool's valuation at time t is given by (cf. Mohan (2022, p. 22))

$$V_{LP}(t) = y_t + x_t\,S_t = y_t + \frac{\tau^2 + 2(1-\tau)}{2(1-\tau)}\,y_t = \frac{(2-\tau)^2}{2(1-\tau)}\,y_t,$$

where we used the mid-price derived from (24). This can be considered a pool valuation from the perspective of liquidity providers, cf. Mohan (2022, p. 23). Assuming that transaction fees are held outside of the liquidity pool after each trade, the liquidity provider's wealth process is

$$V_{LP}(t) = \frac{(2-\tau)^2}{2(1-\tau)}\,L\sqrt{S_t},$$

so that Impermanent Loss can be hedged with appropriate adjustments using the same principles as in Subsections 3.3, 4.1. Further analyses are required if transaction fees remain within the liquidity pool, and will appear elsewhere.

6. Beyond constant product markets

Consider a Constant Function Market whose trading function (cf. 1) corresponds to Cobb-Douglas utility indifference pricing, i.e.

$$\ell((x,y),(\Delta x,\Delta y)) := (x+\Delta x)^\alpha (y+\Delta y)^{1-\alpha}, \quad \alpha \in (0,1); \qquad (25)$$

$\alpha = 1/2$ being the case of a Constant Product Market.

Remark 6.1 Constant Function Markets with trading function (25) belong to the class of geometric mean market makers, cf. Evans (2020).

As before, we define a constant

$$L := x_0^\alpha\, y_0^{1-\alpha}, \qquad (26)$$

where x_0 and y_0 are the initial LP reserves. Since any transaction $(\Delta x, \Delta y) \in \mathbb{R}^2$ must satisfy

$$(x+\Delta x)^\alpha (y+\Delta y)^{1-\alpha} \equiv L,$$

we have

$$\begin{aligned} \Delta y &= \frac{L^{1/(1-\alpha)}}{(x+\Delta x)^{\alpha/(1-\alpha)}} - \frac{L^{1/(1-\alpha)}}{x^{\alpha/(1-\alpha)}} \\ &\approx -L^{1/(1-\alpha)}\frac{\alpha}{1-\alpha}x^{-1/(1-\alpha)}\Delta x. \end{aligned}$$

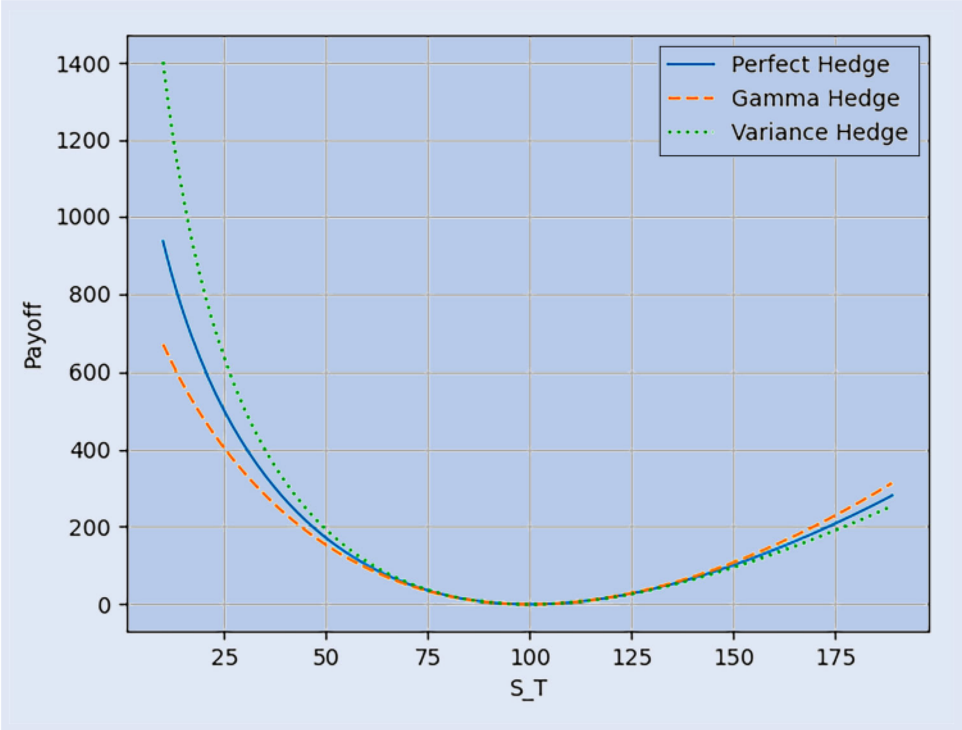

Figure 1. *Terminal Payoff*. Payoffs of the weighted variance swap (solid line termed 'Perfect Hedge'), the gamma swap (dashed), and the variance swap (dotted). $S_0 = 100$.

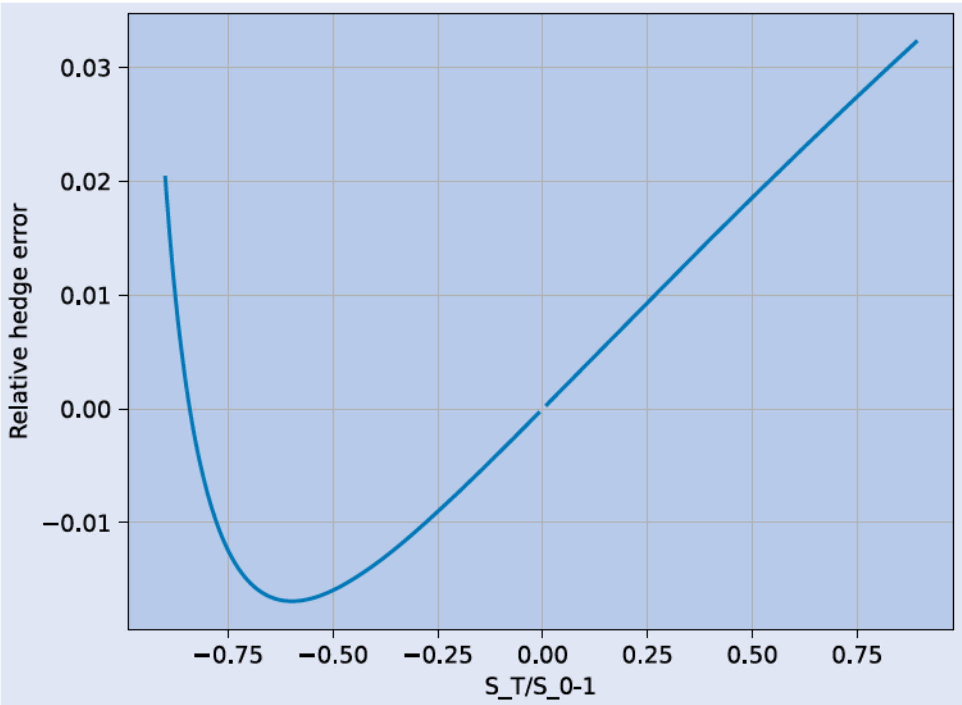

Figure 2. *Hedging Error*. This figure plots the hedge approximation using a variance and gamma swap (portfolio weight $w^* = 0.61$) in terms of relative error as a function of the return $S_T/S_0 - 1$. The hedging error stays below $\pm 3\%$ for the wide range of underlying asset returns from -75% to +75%.

Therefore, the infinitesimal price (exchange ratio) is

$$S = L^{1/(1-\alpha)} \frac{\alpha}{1-\alpha} x^{-1/(1-\alpha)},$$

and this has to coincide with the price in an external market for market viability. Together with (25), we have, at time $t > 0$,

$$x_t = L \left(\frac{\alpha}{1-\alpha} \right)^{1-\alpha} \frac{1}{S_t^{1-\alpha}}, \quad y_t = L \left(\frac{\alpha}{1-\alpha} \right)^{-\alpha} S_t^{\alpha}.$$

At time $t > 0$, the liquidity provider's wealth (in units of the asset Y) that is locked in the LP equals

$$V_{LP}(t) = y_t + x_t S_t = L\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} \frac{1}{1-\alpha} S_t^{\alpha}.$$

Now suppose that S_t is the price process in the external market and the liquidity provider finances her initial pool (x_0, y_0) from the external market at time $t = 0$. Her Impermanent Loss at time T is

$$\begin{aligned} H_T &= V_{BH}(T) - V_{LP}(T) \\ &= (y_0 + x_0 S_T) - L\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} \frac{1}{1-\alpha} S_T^{\alpha} \\ &= -L\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} S_0^{\alpha}\left(\frac{1}{1-\alpha}\left(\frac{S_T}{S_0}\right)^{\alpha} - 1 - \frac{\alpha}{1-\alpha}\frac{S_T}{S_0}\right). \end{aligned}$$

This payoff can be statically hedged by using an European power contract. With the reparametrization

$$\widetilde{L} := L\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha},$$

we have the alternative expression

$$H_T = \widetilde{L} S_0^{\alpha}\left(\left(\frac{S_T}{S_0}\right)^{\alpha} \frac{1}{1-\alpha}\left(\left(\frac{S_T}{S_0}\right)^{1-\alpha} - 1\right) + 1 - \frac{S_T}{S_0}\right). \tag{27}$$

REMARK 6.2 A contract with the payoff given in (27) hedges against Impermanent Loss in a constant-weight geometric mean market maker with two tokens. We refer to Evans (2020) for the wealth process of a general geometric mean market maker (with several tokens and dynamic weights).

Letting $\alpha \to 1$, the limit payoff is

$$\widetilde{L}\left(S_T \log\frac{S_T}{S_0} - S_T + S_0\right),$$

which can be hedged by a static portfolio of the Gamma swap (entropy contract).

With a different reparametrization,

$$\hat{L} = L\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha},$$

we have yet another representation:

$$H_T = -\hat{L} S_0^{\alpha}\left(\frac{1}{\alpha}\left(\left(\frac{S_T}{S_0}\right)^{\alpha} - 1\right) + 1 - \frac{S_T}{S_0}\right).$$

Letting $\alpha \to 0$, the limit payoff is

$$-\hat{L}\left(\log\frac{S_T}{S_0} + 1 - \frac{S_T}{S_0}\right),$$

which can be hedged by a static portfolio of the variance swap (the 'log contract', cf. Neuberger (1994)).

REMARK 6.3 Note that, for the limit $\alpha \to 1$ ($\alpha \to 0$), we keep the reparametrization $\widetilde{L}$ ($\hat{L}$) constant. This means that L diverges in the limit $\alpha \to 1$ ($\alpha \to 0$).

PROPOSITION 6.4 *In a Constant Function Market whose trading function is given by* (25)*, the static hedge against Impermanent Loss* (9) *can be approximated with a variance swap as* $\alpha \to 0$*, and with a Gamma swap as* $\alpha \to 1$.

REMARK 6.5 In Evans (2020, p. 6), the LP's value in the limiting case $\alpha = 1$ ($\alpha = 0$) is interpreted as a buy-and-hold strategy of token X (token Y) exclusively. The difference with our result is that we require the reparametrization $\tilde{L}$ ($\hat{L}$) to remain constant.

We can directly construct utility functions (trading functions) corresponding to gamma swap and variance swap payoffs, respectively, as we show in the following subsections.

6.1. The constant function market whose impermanent loss can be hedged with a gamma swap

Consider the trading function

$$\ell((x, y), (\Delta x, \Delta y)) := x + \Delta x + \log(y + \Delta y)$$

Then, defining $L := x + \log y$, we get

$$\Delta y = e^{L-x-\Delta x} - y = e^{L-x-\Delta x} - e^{L-x} \approx -\Delta x e^{L-x},$$

which implies that $S = e^{L-x} = y$ is the infinitesimal price of asset Y in units of asset X. Therefore,

$$x = L - \log S, \quad y = S, \quad y + xS = S(1 + L - \log S),$$

and so Impermanent Loss at time T is

$$\begin{aligned} H_T &= S_0 + (L - \log S_0)S_T - S_T(1 + L - \log S_T) \\ &= -S_T + S_0 + S_T \log\frac{S_T}{S_0}. \end{aligned}$$

6.2. The constant function market whose impermanent loss can be hedged with a variance swap

Similarly, the trading function

$$\ell((x, y), (\Delta x, \Delta y)) := \log(x + \Delta x) + (y + \Delta y)$$

gives $\Delta y \approx -\frac{\Delta x}{x}$, meaning that $S = \frac{1}{x}$. In this case, with $L := \log x + y$, we have

$$x = \frac{1}{S}, \quad y = L - \log x, \quad y + xS = 1 + L + \log S.$$

Impermanent Loss thus becomes

$$\begin{aligned} H_T &= \left(L + \log S_0 + \frac{S_T}{S_0}\right) - (1 + L + \log S_T) \\ &= \frac{S_T}{S_0} - 1 - \log\frac{S_T}{S_0}, \end{aligned}$$

as noted in Angeris et al. (2021b, Subsection 2.4).

6.3. Constructing the constant function market corresponding to a general payoff

Under general C^1 utility indifference pricing (cf., e.g. Carmona 2009)

$$\ell((x,y),(\Delta x,\Delta y)) := u(x+\Delta x, y+\Delta y)$$

with non-degenerate marginal utility

$$\frac{\partial u}{\partial x} \neq 0, \quad \frac{\partial u}{\partial y} \neq 0,$$

we have

$$\frac{\partial u}{\partial x}(x,y)\Delta x + \frac{\partial u}{\partial y}(x,y)\Delta y \approx 0,$$

which implies the price of Y in units of X in an external market to be

$$S = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}(x,y).$$

The implicit function theorem on the other hand ensures the existence of a C^1 function f such that $u(x,f(x)) = L$ with

$$f'(x) = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}(x,f(x)) = -S.$$

For a reasonably chosen utility function, the reserve x in the LP is a strictly decreasing function of the external price S, and so we assume f' to be strictly increasing, or equivalently, f to be strictly convex hereafter. The LP's value in units of Y is

$$V_{LP} = y + xS = f(x) - xf'(x) = f(x^*(p)) - x^*(p)p,$$

where $p = -S$ and $x^*(p) = (f')^{-1}(p)$. We then conclude

$$V_{LP} = -f^*(-S),$$

where $f^*(p)$ is the Legendre transform of f:

$$f^*(p) = \sup_x \{px - f(x)\}.$$

In particular, we find that V_{LP} is a concave function of S. Further, since $(f^*)'(p) = x^*(p)$, V_{LP} should be nondecreasing. The property $(-g)^{**} = -g$ for a concave function g gives the following.

Proposition 6.6 *Any nondecreasing, strictly concave C^1 payoff $V_{LP}(S) = g(-S)$ of S on $(0,\infty)$ can be replicated in a Constant Function Market with a utility function u such that $u(x,f(x))$ is constant, where f is the Legendre transform of the nondecreasing strictly convex C^1 function $-g$ on $(-\infty,0)$.*

Remark 6.7 In Angeris *et al.* (2021b, Subsection 1.1), the authors link Constant Function Markets' trading functions and payoff profiles via the Fenchel transform. We consider the reasoning put forth herein more straightforward.

Example 6.8 For a Constant Product Market, whose trading function is given by (5), the expressions above reduce to $S = y/x$, $f(x) = 1/x$, and $f^*(p) = -2\sqrt{-p}$, so that $V_{LP} = -f^*(-S) = 2\sqrt{S}$, as in (11).

7. Conclusion

In this paper, we analyzed the connection between Constant Function Markets and variance swaps and gamma swaps, which are important and extensively studied volatility products in traditional finance ('TradFi'). This link between DeFi and TradFi is established by the hedging of Impermanent Loss that a liquidity provider suffers vis-à-vis a HODLer. In particular, we showed that Impermanent Loss in a Constant Product Market can be hedged statically with a weighted variance swap of order 1/2. We furthermore derived the Constant Function Market's trading functions such that Impermanent Loss can be hedged with variance swaps or gamma swaps. These results are similar in spirit to the research of Angeris *et al.* (2021a, 2021b) on designing Constant Function Markets whose trading functions are specified in such a way as to guarantee the liquidity provider a certain payoff, such as that of a covered call or a portfolio with equal weights.

We strongly believe that the intersection between TradFi and DeFi highlighted herein offers a rich field of research for experts in quantitative finance, and we hope that its cross-fertilization will contribute to the maturing process of Decentralized Finance.

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Appendices

Appendix 1. Formulae under the Heston model

Under the Heston model

$$\frac{\mathrm{d}S_t}{S_t} = \sqrt{v_t}\left\{\rho\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}W_t^{\perp}\right\},$$

$$\mathrm{d}v_t = \kappa(\theta - v_t)\,\mathrm{d}t + \eta\sqrt{v_t}\,\mathrm{d}W_t,$$

where $(W, W^{\perp})$ is a two-dimensional standard Brownian motion, we know explicit formulae for the variance swap and gamma swap prices Υ_0 and $\mathcal{E}_0$ exist:

$$\Upsilon_0 = \theta T + (v_0 - \theta)\frac{1 - e^{-\kappa T}}{\kappa},$$

$$\mathcal{E}_0 = \frac{\kappa}{\kappa'}\theta T + \left(v_0 - \frac{\kappa}{\kappa'}\theta\right)\frac{1 - e^{-\kappa' T}}{\kappa'},$$

where $\kappa' = \kappa - \eta\rho$. The two prices coincide when $\rho = 0$. More generally, they coincide under any stochastic volatility model with no leverage effect. See Fukasawa (2014) for the details.

An involved but nevertheless explicit expression for the Impermanent Loss hedge (12) can be inferred directly from del Baño Rollin et al. (2010, Subsection 2.1); cf. also Macovschi and Quittard-Pinon (2006, Section II). The authors' approach in del Baño Rollin *et al.* (2010) is based on the characteristic function (in their notation, $X_t = \log S_t$, $u = 1/2$, $a = \kappa$, $b = \theta$, $c = \eta$, $\mu = 0$).

Appendix 2. Computing the hedge premium from options data

We use options data as of 25 March, 2022 as displayed in Table A1. Assuming that a liquidity provider establishes an LP containing $x_0 = 1$ BTC and $y_0 = 42,955$ USD (so that both amounts have the same dollar value), Formula (15) yields that the value of her static hedge against Impermanent Loss is approximately USD 178.84. In fact, from this, we can back out the implied volatility using the approximation of the fee (cf. (14)), $y_0\sigma^2 T/4$, which yields $\sigma \approx 65.90\%$. Note that this is close to the implied volatility based on six months of data, which stood roughly at 65.57% as of 25 March 2022, cf. The Block Crypto.

Table A1. *Option Prices*. Vanilla BTCUSD calls and puts with two-weeks maturity, on 25 March 2022 (ATM = 42,955). We thank SEBA Bank for kindly making these data available to us.

Calls		Puts	
Premium	Strike	Premium	Strike
2072	43,000	2095	43,000
1841	43,500	1849	42,500
1629	44,000	1623	42,000
1437	44,500	1419	41,500
1263	45,000	1235	41,000
1107	45,500	1072	40,500
968	46,000	928	40,000
845	46,500	798	39,500
733	47,000	684	39,000
636	47,500	584	38,500
550	48,000	499	38,000
476	48,500	427	37,500
411	49,000	367	37,000
355	49,500	318	36,500
307	50,000	275	36,000
267	50,500	236	35,500
232	51,000	202	35,000
202	51,500	174	34,500
176	52,000	151	34,000
154	52,500	131	33,500
134	53,000	114	33,000
118	53,500	100	32,500
104	54,000	88	32,000
92	54,500	77	31,500
81	55,000	68	31,000
72	55,500	60	30,500
64	56,000	53	30,000
57	56,500	47	29,500
52	57,000	42	29,000
46	57,500	37	28,500
42	58,000	33	28,000
37	58,500	29	27,500