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Research paper

Analytical solution of an Ill-posed system of nonlinear ODE's

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ABSTRACT

In this paper, we discuss a solution strategy for an overdetermined ODE system that uses elliptic functions. In particular, we show how an apparently ill-posed ODE system can be made amenable to treatment by Jacobian elliptic functions by introducing additional free parameters and still obtain a solution that corresponds well to the physically expected behaviour.

1. Introduction

The mechanical system *cubli* was first presented in [1,2]. It is a mechanical-dynamic system that can move and balance. As explained in [1], inverted pendulum systems already have a long history (for an overview of the history of the research on inverted pendula see e.g. [3] or [4]), but the development of new algorithms for the control of such systems is still an active area of research, especially since such systems serve as benchmark systems for new control algorithms. The purpose of the cubli system is therefore not so much its direct use in an industrial context, but rather as an "inexpensive, open source test-bed with a relatively small footprint for research and education in estimation and control" [1]. Similar projects were subsequently launched, e.g. the "Kuutio" [5]; the *cubli* has very recently also been complemented by the "One-Wheel Cubli" [6].

In [7,8], an iterative method is presented to let the system "jump up" from its flat position. Since this system has rather weak motors, this jumping-up must be done by slowing down the fast-spinning wheels with mechanical breaks. The iterative process manipulates a subsequent action of the motors to achieve the upright unstable equilibrium state. The system has 2 degrees of freedom, the rotation of the rigid body around its corner and the rotation of the flywheel, described by the two coordinates ϕ_1 and ϕ_2. Note, that the angle ϕ_2 is described in the fixed coordinate system of the body (see Fig. 1). Moreover, the system depends on 3 parameters which are generally difficult to determine experimentally.

If we now consider a lifting process, we assume that at the beginning of the process the cuboid is at rest and that the flywheel has a certain angular velocity, and that at the end of the process both are at rest. From a mathematical point of view, these restrictions lead to an overdetermined ODE system, so that in general one cannot expect a solution of the system satisfying all restrictions for an arbitrarily given external moment acting on the cuboid. Usually, the system is investigated with iterative (as mentioned above) or numerical methods (see also [9] or [10]), or then from the perspective of optimal control of ODE's. For recent control-theoretic perspectives on the model, we refer to [11–13]; there is also a vast literature on inverted pendula as control-theory reference models, see e.g. the overviews provided by [14,15], but the topic has also seen some (very) recent advances (e.g. [16–23]).

In this paper, we will show that one can nevertheless solve the system analytically by assuming a special form of the external momentum. The solution will involve elliptic Jacobi functions. While the assumed ansatz for the external momentum does not seem to be sufficiently motivated by the real-world situation from which the system originates, the three free parameters of the system mentioned above can subsequently be adapted to obtain a member of our 3-parameter family of analytic solutions that corresponds

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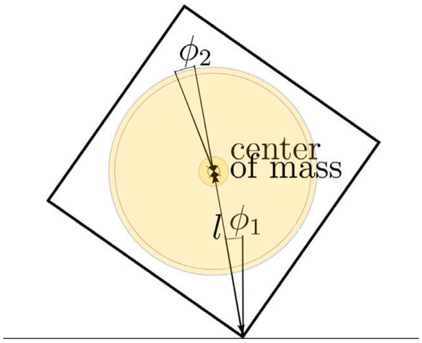

Fig. 1. The 2 degrees of freedom are described by ϕ_1 and ϕ_2.

to the results obtained in laboratory experiments. By being able to fit these parameters to real results, we can combine the clarity of an analytically closed solution with the usefulness in describing a real-world situation with a system that at first sight seems ill-posed.

The search for exact solutions to ODEs or systems of ODEs remains an active area of research despite the enormous advances in computational power that enable increasingly sophisticated numerical methods. The availability of exact solutions in certain special cases enables and facilitates a structural understanding of these selected problems that is difficult to achieve with numerical and/or approximate methods; moreover, systems with exact solutions can also serve as test cases for verifying the accuracy and precision of numerical and other approximate methods (we note, however, that the superiority of analytical over numerical solutions has also been questioned [24]). As can be seen from the (incomplete) list of references, exact solutions have recently been studied and discovered in a variety of areas such as Riccati-type equations [25], more general autonomous nonlinear equations [26], nonlinear ODE's with exact solutions expressable via Weierstrass functions [27], and also in the understanding of the dynamics of neural networks [28] or more general network ODE systems with input excitations [29] (which has to be distinguished from using ODE solvers for dealing with neural networks [30] or even using neural networks to solve ODE's, see e.g. [31] and the references therein); for many additional references see the recent review article [32]. Note that in addition to the initial value problems usually considered, there have also been advances in research on boundary value problems for ODEs, see e.g. [33]. Recently, efforts have been made to use symmetry properties of differential equations to obtain explicit solutions, mainly for PDEs, but also for certain oscillators modelled by nonlinear ODEs [34]. In a related but slightly different approach, nonlinear ODE systems with stable equilibria (which mainly occur in mathematical biology) are approximately solved by analytical functions [35].

The value of the analytical solution also lies in its application for educational purposes; however, beyond that, it provides a reference solution for validating numerical methods developed for solving similar problems. In our view, the analytical solution is extremely valuable, for example, for the validation of high-fidelity structural models that can be explicitly integrated in time. Furthermore, since the solution is available in parameter space, it also enables the validation of surrogate modelling methods. In particular, one should therefore be able to successfully match them with appropriate data.

Parametrized differential equation systems such as *cubli* are widely used in engineering to model time-dependent processes (for a classical introduction to this broad topic see e.g. [36] or [37]). The corresponding parameters are often not given a priori in a natural way and have to be estimated a posteriori with the help of empirical data. If no analytical solutions are available for the parametrized differential equation systems, the a posteriori determination of the parameters can be very difficult or very costly, as the various previous approaches for the cubli system have shown. This is generally the case when the model is non-linear in nature. In this case, an extensive toolbox of numerical methods is available today to estimate the unknown parameters (for an overview of this broad field, we recommend [38–40]). The initial situation is greatly simplified when complete analytical solutions for the nonlinear differential equations are known. This adds additional value to our calculations. However, even in this case the estimation of the parameters remains challenging, but here the classical methods of nonlinear regression analysis are directly applicable (for this extensive topic, see also [38–40]). This partly explains the importance of analytical solutions of parametrized differential equation systems, even if they involve functions that are not classically called elementary, such as the Lambertian W-function (see [41] for a recent application and [42] for a compilation of models).

In this paper, an analytical solution to a classical benchmark problem from dynamical systems theory is found using elliptic Jacobian functions. This result is remarkable because the nonlinear differential equation system of *cubli* was not previously considered to be analytically treatable. This analytical solution contains three parameters and one should therefore be able to successfully match it with appropriate data. Moreover, it fits into the framework of current engineering research methods in which analytical and numerical methods are pragmatically mixed to achieve the desired results (admittedly arbitrarily focused on examples from aerodynamics see e.g. [43–46]).

Limitations of the study: The issue of stabilization of the proposed exact solution in the final position is not addressed in the study. Preliminary investigations (analytical and experimental) indicate that stability is not sufficiently guaranteed under perturbed initial conditions, and it is the subject of our ongoing investigations to extend/modify the proposed solution method to also address the stability issue.

Overview of the paper In Section 2 we formulate the problem and outline our solution strategy. In Section 3 we formulate the main result and give its proof, up to the determination of the various parameters that appear in the result, which we then determine in Section 4; i.e. Section 4 consists entirely of the proof of a theorem stated in Section 3 on the values of the relevant parameters. In Section 5 we compute our solution numerically and discuss its relation to experimentally available results. Section 6 concludes the paper.

2. Mathematical formulation

The dynamics can easily be derived from the principle of angular momentum:

$$\Theta_1 \ddot{\phi}_1 = m\, g\, \sin(\phi_1) - M_{\mathrm{Mot}}(t) \tag{1}$$

$$\Theta_2 \ddot{\tilde{\phi}}_2 = M_{\mathrm{Mot}}(t) \tag{2}$$

Here m is the total mass of the cuboid and the flywheel, Θ_1 and Θ_2 are the moments of inertia of the cuboid and flywheel, $\tilde{\phi}_2$ is the angle of the flywheel in the world frame, hence $\tilde{\phi}_2 = \phi_1 + \phi_2$. We use the angle ϕ_2 in the body frame from now, since that is the value that can be measured directly.

The constant values can be summarized to two values α and β, as the following normed setting shows: We can reformulate the dynamics described by (1)–(2) by the two coupled ordinary differential equations

$$\ddot{\phi}_1(t) = \alpha \cdot \sin(\phi_1(t)) - f(t) \tag{3}$$

$$\ddot{\phi}_1(t) + \ddot{\phi}_2(t) = \beta \cdot f(t) \tag{4}$$

for the two unknown functions

$$\phi_1(t) \geq -\frac{\pi}{4}, \quad \phi_2(t) \geq 0,$$

with the seven boundary conditions

$$\phi_1(0) = -\frac{\pi}{4},\ \phi_1(\tau) = 0,\ \dot{\phi}_1(0) = 0,\ \dot{\phi}_1(\tau) = 0 \tag{5}$$

as well as

$$\phi_2(0) = 0,\ \dot{\phi}_2(0) = \omega \tag{6}$$

and

$$\dot{\phi}_2(\tau) = 0. \tag{7}$$

According to the condition $\phi_1(\tau) = 0$ given by (5), τ is the ending time of the lift-up procedure described in the previous section, and ω is an initial angular velocity. These boundary conditions follow from the physical restrictions on the lift-up process described in Section 1.

Note that when comparing the dimensionless version (3)–(4) of our system with the original physical version (1)–(2), the various constants are related by $\alpha = \frac{mg}{\Theta_1}$ and $\beta = \frac{\Theta_1}{\Theta_2}$. In addition, note that our moment function $f(t)$ appearing in (3)–(4) is related to the physical moment function $M_{\mathrm{Mot}}(t)$ appearing in (1)–(2) by $f(t) = \frac{M_{\mathrm{Mot}}(t)}{\Theta_1}$. A sample computation with numerical values of the constants will be given in Section 5.

Since a system of two second order ODEs in general admits only 4 additional conditions, the system (3)–(4) is overdetermined, and we may only hope for solutions if the constant τ and the function $f(t)$ are chosen appropriately.

The system (3)–(4) is half decoupled in the sense that Eq. (3) is an ODE for $\phi_1(t)$ only. Therefore we firstly focus on (3) together with the additional conditions (5). Once the function ϕ_1 has been determined Eq. (4) becomes an elementary ODE of second degree for the function ϕ_2: For given ϕ_1, two additional integration steps suffice to determine the function ϕ_2. The additional conditions (6) will ensure the uniqueness of the solution function ϕ_2. In the following we will therefore focus on the functions ϕ_1 and f, since it is in the conditions for ϕ_1 where the illposedness of the problem comes into play.

Finally, the condition (7) on ϕ_2 is equivalent to an additional condition on the function $f(t)$. Indeed, we have on the one side, by (6) and (7),

$$\int_0^\tau \ddot{\phi}_2(t)\mathrm{d}t = \dot{\phi}_2(\tau) - \dot{\phi}_2(0) = -\omega$$

and on the other side, by (4) and (5),

$$\int_0^\tau \ddot{\phi}_2(t)\mathrm{d}t = \int_0^\tau \left(\beta \cdot f(t) - \ddot{\phi}_1(t)\right) \mathrm{d}t = \beta \cdot \int_0^\tau f(t)\mathrm{d}t.$$

Therefore the function $f(t)$ has to fulfil

$$\int_0^\tau f(t)\mathrm{d}t = -\frac{\omega}{\beta}. \tag{8}$$

Condition (8) is therefore equivalent to condition (7), and in the sequel we will replace condition (7) by condition (8).

Solution strategy As stated in the previous discussion, the problem (3)–(7) is overdetermined if the parameter τ and the external moment function $f(t)$ are taken as given, since we are dealing with two second-order ODE's together with 7 boundary conditions. The overdeterminacy becomes even clearer if we restrict the discussion to a problem for ϕ_1, namely if we take the (ϕ_2-independent) ODE (3) for ϕ_1 together with the 4 boundary conditions (5) for ϕ_1 and the integral condition (8) on $f(t)$. Recall that condition (8) facilitates the determination of ϕ_2 with 2 simple integrations once we have solved the problem for ϕ_1. Therefore, we consider it desirable to concentrate all the overdeterminacy of the original given physical problem in the mathematical problem of finding a solution of a boundary value problem for ϕ_1 with 5 constraints, instead of the 2 constraints normally expected. It is clear that the 3 "excess" constraints can be reduced to 2 "excess" constraints if we allow some freedom in the choice of the parameter τ.

Our main idea now is to introduce 2 additional parameters by prescribing a very special form of the external function $f(t)$, namely that it can be written as a trigonometric polynomial *of* $\phi_1(t)$, i.e. in the form $f(t) = u \cdot \sin(\phi_1(t)) + v \cdot \cos(\phi_1(t))$. The 2 additional parameters u and v that occur in this representation of $f(t)$ are chosen in such a way that the problem admits a unique solution for ϕ_1 and is amenable to a treatment by elliptic functions.

Considering the physical origin of the problem, it may seem strange that the external momentum $f(t)$ depends on the solution function ϕ_1 and not vice versa. However, this reversal of dependence seems to be justified by the fact that the solution we can find under this rather unusual ansatz behaves qualitatively exactly as can be observed in the original laboratory experiment when "typical" momentum functions are used. Moreover, since the momentum function in the laboratory experiment can also be prescribed and mechanically implemented to a considerable extent, our restriction of the momentum function is practically much less relevant than would be expected from a purely mathematical point of view. From the mathematical side, the fact that ϕ_1 is a monotonic function allows one to use ϕ_1 as a parameter function for $f(t)$. In this case of this restriction of $f(t)$ with these additional parameters u and v, the solution ϕ_1 can be found in terms of elliptic functions, and the parameters u and v are determined by the boundary conditions (5) on ϕ_1, as well as τ by the condition (8).

The core idea of giving the moment function $f(t)$ a form that allows the use of the toolbox of elliptic functions is, in our view, not primarily justified by the fact that the problem becomes amenable to an exact analytical treatment, but rather by the fact that these constraints, which may at first seem physically unmotivated, nevertheless lead to a result that is good for a theoretical understanding of the laboratory experiment that forms the origin of our seemingly overdetermined mathematical problem.

3. Analytical solution by elliptic functions

As mentioned before, here we make for the moment function f an ansatz which is motivated by making the system accessible to the machinery of elliptic functions but which is, by introducing sufficiently many parameters, at the same time flexible enough to account for all given conditions, and which leads to solutions closely resembling the laboratory reality, namely that it can be expressed as a trigonometric polynomial of ϕ_1, which then leads to a closed solution for ϕ_1 by elliptic functions. I.e., we assume that $f(t)$ is of the form

$$f(t) = u \cdot \sin(\phi_1(t)) + v \cdot \cos(\phi_1(t)) \tag{9}$$

for parameters u, v yet to be determined. The ODE (3) for ϕ_1, namely

$$\ddot{\phi}_1(t) = \alpha \cdot \sin(\phi_1(t)) - f(t)$$

then reads

$$\ddot{\phi}_1(t) = \alpha \cdot \sin(\phi_1(t)) - (u \cdot \sin(\phi_1(t)) + v \cdot \cos(\phi_1(t)))$$

It is well known (see e.g. [47–49]) that 2nd order ODE's of the form

$$y'' = f(y)$$

can be integrated to the separable ODE

$$y' = \pm\sqrt{2F(y) + C}, \tag{10}$$

where $F(\cdot)$ is an antiderivative of $f(\cdot)$. After separation of variables, one obtains an implicit solution for the original ODE, which can only be made explicit in case the corresponding integral is elementary. The purpose of our paper is to show that by making accessible the original system (3)–(7) to this procedure by the assumption (9), one can cope with the overdeterminedness of the problem described in the previous section by assuming sufficiently many parameters and then obtain a solution which – for typical values of the parameters of the original system – matches remarkably well what can be observed experimentally.

Remark. The concept of closed solution is not well-defined in full generality; terminology needs a clarification. In Appendix we give a list a functions which form the basis of what we henceforth call a closed solution. I.e. any solution of the initial problem that can be expressed with the help of one of the functions in the list contained in Appendix will be called closed throughout this paper.

Our main result is the following.

Theorem 3.1. *Let $\alpha, \beta, \omega > 0$ be fixed. Then there exist constants $u, v, \tau \in \mathbb{R}$ depending only on α, β, ω such that the initial–boundary-integral value problem consisting of the ODE*

$$\ddot{\phi}_1(t) = \alpha \cdot \sin(\phi_1(t)) - (u \cdot \sin(\phi_1(t)) + v \cdot \cos(\phi_1(t))), \tag{11}$$

the initial–boundary conditions

$$\phi_1(0) = -\frac{\pi}{4} \tag{12}$$

$$\phi_1(\tau) = 0 \tag{13}$$

$$\dot{\phi}_1(0) = 0 \tag{14}$$

$$\dot{\phi}_1(\tau) = 0 \tag{15}$$

and the integral condition

$$\int_0^\tau (u \cdot \sin(\phi_1(t)) + v \cdot \cos(\phi_1(t)))\mathrm{d}t = -\frac{\omega}{\beta} \tag{16}$$

admits a unique solution. It is given by

$$\phi_1(t) = 2 \cdot \arcsin\left(\mathrm{sn}\left(\kappa_2 \cdot t - \kappa_0 \middle| \sin^{-2}\left(\frac{\pi}{16}\right)\right)\right) - \frac{\pi}{8} \tag{17}$$

where κ_0 and κ_2 also depend only on α, β, ω. Here $sn(u|m)$ is one of the Jacobian elliptic functions which are reviewed in Appendix, see (51).

Proof. The proof of Theorem 3.1 principally consists of two integration steps, following the general scheme for ODE's of the form $y'' = f(y)$, but adapted to the present situation of coping with the overdeterminedness: In a first integration, we reduce the second-order ODE (11) to a separable first-order ODE, which we then integrate in a second step by using elliptic functions.

Note that we overcome the overdeterminedness of the problem statement (5 conditions instead of 2 conditions as usual for a second-order ODE) by introducing 3 parameters already in the problem statement (u, v, and τ), which we choose in such a way that the problem admits a solution. The uniqueness of the solution follows by the constructive nature of our proof. It is evident from the problem statement that because of the mixed nature of the problem (initial and boundary values, integral condition, overdeterminedness), we cannot use existing standard results from ODE theory.

First integration step: Note that we can rewrite the ODE (11) as

$$\ddot{\phi}_1 = (\alpha - u) \cdot \sin(\phi_1) - v \cdot \cos(\phi_1), \tag{18}$$

A first integral of Eq. (18) is

$$\frac{\dot{\phi}_1^2}{2} = (u - \alpha) \cdot \cos(\phi_1) - v \cdot \sin(\phi_1) + C. \tag{19}$$

For $t = \tau$ the boundary conditions (14)–(15) imply

$$C = -(u - \alpha). \tag{20}$$

For $t = 0$ conditions (12)–(13) imply, already using (20),

$$v = (u - \alpha) \cdot (\sqrt{2} - 1). \tag{21}$$

Therefore the first integral (19) reads

$$\dot{\phi}_1 = \sqrt{2 \cdot (u - \alpha)} \cdot \sqrt{\cos(\phi_1) + \left(1 - \sqrt{2}\right) \cdot \sin(\phi_1) - 1}, \tag{22}$$

i.e. we have obtained a separable first order ODE for ϕ_1, in the spirit of (10). Setting

$$\delta = \sqrt{4 - 2 \cdot \sqrt{2}} \tag{23}$$

the separable ODE (22) reads

$$\dot{\phi}_1 = \sqrt{2 \cdot (u - \alpha)} \cdot \sqrt{\delta \cdot \cos\left(\phi_1 + \frac{\pi}{8}\right) - 1}. \tag{24}$$

We now reformulate (24) and the boundary conditions (5) in terms of the auxiliary variable

$$\psi = \phi_1 + \frac{\pi}{8} \tag{25}$$

and obtain from (24)

$$\dot{\psi} = \sqrt{2 \cdot (u - \alpha)} \cdot \sqrt{\delta \cdot \cos(\psi) - 1} \tag{26}$$

together with the boundary conditions for ψ, which are taken from (12)–(15) and now read

$$\psi(0) = -\frac{\pi}{8},\ \psi(\tau) = \frac{\pi}{8},\ \dot{\psi}(0) = 0,\ \dot{\psi}(\tau) = 0.$$

Second integration step: By separating variables in (26) and using the identity $\cos(\psi) = 1 - 2\sin^2\left(\frac{\psi}{2}\right)$, we get

$$\int_{-\frac{\pi}{8}}^{\psi(t)} \frac{\mathrm{d}\psi}{\sqrt{\delta \cdot \left(1 - 2 \cdot \sin^2\left(\frac{\psi}{2}\right)\right) - 1}} = \sqrt{2 \cdot (u - \alpha)} \cdot t,$$

or finally, making the substitution $\psi \to \frac{\psi}{2}$,

$$\int_{-\frac{\pi}{16}}^{\frac{\psi(t)}{2}} \frac{\mathrm{d}\psi}{\sqrt{1 - m \cdot \sin(\psi)^2}} = \sqrt{\delta - 1} \cdot \sqrt{2 \cdot (u - \alpha)} \cdot \frac{t}{2} \tag{27}$$

with

$$m = \frac{2 \cdot \delta}{\delta - 1} = \sin^{-2}\left(\frac{\pi}{16}\right) \tag{28}$$

Using the elliptic integral $F(\phi|m)$, see (50), Eq. (27) may be written in a more compact way as

$$F\left(\frac{\psi(t)}{2}\middle| m\right) = \kappa_2(u) \cdot t - \kappa_0, \tag{29}$$

with

$$\kappa_2(u) = \frac{\sqrt{\delta - 1} \cdot \sqrt{2 \cdot (u - \alpha)}}{2} \tag{30}$$

$$\begin{aligned} \kappa_0 &= F\left(\frac{\pi}{16}\middle| m\right) \\ &= F\left(\frac{\pi}{16}\middle| \sin^{-2}\left(\frac{\pi}{16}\right)\right) \approx 0.3094, \end{aligned} \tag{31}$$

using (28) and evaluating the elliptic integral (50). By solving (29) for $\psi(t)$ we obtain, using the Jacobian elliptic functions reviewed in (51),

$$\psi(t) = 2 \cdot \arcsin(\mathrm{sn}(\kappa_2(u) \cdot t - \kappa_0 | m)) \tag{32}$$

and therefore, by (25),

$$\phi_1(t) = 2 \cdot \arcsin(\mathrm{sn}(\kappa_2(u) \cdot t - \kappa_0 | m)) - \frac{\pi}{8} \tag{33}$$

This gives the claimed solution formula (17) up to the determination of parameters. Note that m and κ_0 are given by (28) and (31), respectively. In addition, by (30), κ_2 is given by

$$\kappa_2 = \sqrt{\frac{\delta - 1}{2}} \cdot \sqrt{u - \alpha} \tag{34}$$

It thus remains to determine the constants u, v, w stated in Theorem 3.1.

Proposition 3.2. *The constants u, v, and w appearing in Theorem 3.1 are given by*

$$u = C_u \cdot \frac{\alpha^2 \beta^2}{\omega^2} + \alpha \qquad v = C_v \cdot \frac{\alpha^2 \beta^2}{\omega^2} \qquad \tau = C_\tau \cdot \frac{\omega}{\alpha \beta} \tag{35}$$

where C_u, C_v, and C_τ do not depend on α, β, ω and can be determined purely analytically.

Assuming Proposition 3.2, this shows that a solution of the boundary–initial-integral value problem stated in Theorem 3.1 must be of the form (17), with κ_0, κ_2 given by (31) and (34), as well as u, v, and τ given (35). The uniqueness statement of Theorem 3.1 is thus proven. Conversely, to prove the existence statement, one just takes the solution formula (17) and checks that all claimed properties are fulfilled. □

4. Analytical computations of the constants

Here we prove Proposition 3.2 on the value of the parameters used in the solution formula of our initial–boundary-integral value theorem.

Proof of Proposition 3.2. As described in the introduction and as given by (5), the parameter τ is by definition the first positive zero of ϕ_1, i.e. the smallest positive solution of $\phi_1(\tau) = 0$. Namely, substituting our preliminary solution formula (33) into $\phi_1(\tau) = 0$, we get, recalling the formula (31) for κ_0 and the definition (51) of the Jacobian elliptic function $\mathrm{sn}(u|m)$,

$$\kappa_2(u) \cdot \tau(u) - \kappa_0 = \kappa_0 \tag{36}$$

and therefore

$$\tau(u) = \frac{2 \cdot \kappa_0}{\kappa_2(u)} \tag{37}$$

This gives τ as a function of $\kappa_2(u)$; note that v is already known as a function of u, cf. (21). To determine u and then $\kappa_2(u)$ by (30), we first rewrite the integrand $u \cdot \sin(\phi_1(t)) + v \cdot \cos(\phi_1(t))$ appearing in the integral condition (16) in terms of $\psi(t) = \phi_1(t) + \frac{\pi}{8}$ and get (plugging (21) into the expression)

$$\begin{aligned} u \cdot \sin(\phi_1(t)) + v \cdot \cos(\phi_1(t)) &= u \cdot \sin\left(\psi(t) - \frac{\pi}{8}\right) \\ &\quad +(u-\alpha) \cdot (\sqrt{2}-1) \cdot \cos\left(\psi(t) - \frac{\pi}{8}\right) \\ &= \lambda_1(u) \cdot \sin\left(\frac{\psi(t)}{2}\right) \cdot \cos\left(\frac{\psi(t)}{2}\right) \\ &\quad -\lambda_2 \cdot \sin^2\left(\frac{\psi(t)}{2}\right) + \frac{\lambda_2}{2} \end{aligned} \tag{38}$$

with

$$\lambda_1(u) = 2 \cdot \left(u \cdot \cos\left(\frac{\pi}{8}\right) + (u-\alpha) \cdot (\sqrt{2}-1) \cdot \sin\left(\frac{\pi}{8}\right)\right)$$

and

$$\begin{aligned} \lambda_2 &= 2 \cdot \left(-u \cdot \sin\left(\frac{\pi}{8}\right) + (u-\alpha) \cdot (\sqrt{2}-1) \cdot \cos\left(\frac{\pi}{8}\right)\right) \\ &= -2\alpha(\sqrt{2}-1)\cos\left(\frac{\pi}{8}\right). \end{aligned} \tag{39}$$

Now, by our solution formula (32) for $\psi(t)$, i.e.

$$\psi(t) = 2 \cdot \arcsin(\mathrm{sn}(\kappa_2(u) \cdot t - \kappa_0|m))$$

with $m = \sin^{-2}\left(\frac{\pi}{16}\right)$ by (28) and again using the definition (51) of the Jacobian elliptic functions, the expression (38) can be recast as

$$\begin{aligned} u \cdot \sin(\phi_1(t)) + v \cdot \cos(\phi_1(t)) &= \lambda_1(u) \cdot \mathrm{sn}(\kappa_2(u) \cdot t - \kappa_0|m) \cdot \mathrm{cn}(\kappa_2(u) \cdot t - \kappa_0|m) \\ &\quad - \lambda_2 \cdot \mathrm{sn}^2(\kappa_2(u) \cdot t - \kappa_0|m) + \frac{\lambda_2}{2} \end{aligned} \tag{40}$$

We now use the integral condition (16), i.e.

$$\int_0^\tau (u \cdot \sin(\phi_1(t)) + v \cdot \cos(\phi_1(t)))\mathrm{d}t = -\frac{\omega}{\beta} \tag{41}$$

By plugging the right hand side of (40) into the integral on the left hand side of (41) and using the substitution $z = \kappa_2 t - \kappa_0$ (thereby using (36) for the treatment of the upper integration limit), we obtain

$$\lambda_2 \cdot (-\eta + \kappa_0) = \frac{\omega}{\beta} \cdot \kappa_2(u) \tag{42}$$

with, by (31) and (54),

$$\eta = \int_{-\kappa_0}^{\kappa_0} \mathrm{sn}(z|m)^2 \mathrm{d}z = -2 \cdot \frac{E(\kappa_0|m)}{m} + 2 \cdot \frac{\kappa_0}{m} \tag{43}$$

Note that by (52) and (53)

$$\int_{-\kappa_0}^{\kappa_0} \mathrm{sn}(z|m) \cdot \mathrm{cn}(z|m)\mathrm{d}z = 0.$$

Next, we observe that by (30) we can rewrite (42) as

$$\lambda_2 \cdot (\kappa_0 - \eta) = -\frac{\omega}{\beta} \cdot \frac{\sqrt{\delta - 1}\sqrt{u-\alpha}}{2}$$

This can be solved for u, and we obtain, using the formula (39) for λ_2,

$$u = 4\alpha^2\left(3 - 2\sqrt{2}\right)\cos^2\left(\frac{\pi}{8}\right) \cdot \frac{2\beta^2(\kappa_0 - \eta)^2}{\omega^2(\delta - 1)} + \alpha$$

$$= \underbrace{\left(4\left(3-2\sqrt{2}\right)\cos^2\left(\frac{\pi}{8}\right)\cdot\frac{2(\kappa_0-\eta)^2}{\delta-1}\right)}_{=:C_u}\cdot\frac{\alpha^2\beta^2}{\omega^2}+\alpha \tag{44}$$

and for v, using (21),

$$v = (u-\alpha)\cdot(\sqrt{2}-1) = \underbrace{(\sqrt{2}-1)C_u}_{=:C_v}\cdot\frac{\alpha^2\beta^2}{\omega^2} \tag{45}$$

Concerning τ, by plugging $\kappa_2 = \sqrt{\frac{\delta-1}{2}}\cdot\sqrt{u-\alpha}$ (see (30)) into $\tau = \frac{2\cdot\kappa_0}{\kappa_2}$ (see (37)), we obtain

$$\tau = \frac{2\cdot\kappa_0}{\kappa_2} = \underbrace{\frac{2\sqrt{2}\cdot\kappa_0}{\sqrt{\delta-1}\sqrt{C_u}}}_{=:C_\tau}\cdot\frac{\omega}{\alpha\beta} \tag{46}$$

From (44), (45), and (46), the formulas (35) and therefore Proposition 3.2 follow. □

We finally compute the numerical values of the various constants in order to get a realistic assessment of our approach which can be compared to what is observed experimentally. Plugging

$$\begin{aligned}\kappa_0 &= F\left(\frac{\pi}{16}\middle| m\right) \approx 0.3094\\ \delta &= \sqrt{4-2\cdot\sqrt{2}} \approx 1.0824\\ \eta &= -2\cdot\frac{E(\kappa_0|m)}{m}+2\cdot\frac{\kappa_0}{m} \approx 0.011\end{aligned}$$

(see (31), (23), (43)) into the formulas for u and v, we get

$$C_u \approx 1.266, \qquad C_v \approx 0.524, \qquad C_\tau \approx 2.717. \tag{47}$$

Concerning κ_0 and κ_2, note that by (31), $\kappa_0 \approx 0.3094$; by (34) and (44), we have

$$\kappa_2 = \sqrt{\frac{\delta-1}{2}}\cdot\sqrt{u-\alpha} = \sqrt{\frac{\delta-1}{2}}\cdot\sqrt{C_u}\cdot\frac{\alpha\beta}{\omega} \approx 0.2277\cdot\frac{\alpha\beta}{\omega} \tag{48}$$

5. Numerical discussion

In order to demonstrate that the statement of Theorem 3.1 leads to a physically realistic solution of the initial value problem originally stated, we numerically compute the constants u, v, ω, whose unique existence we have established in the theorem, and then plot the formulas for the external momentum $f(t)$ and the solution function $\phi_1(t)$.

Typical values of the constants α, β, and ω, which were used in the corresponding laboratory experiment, are

$$\alpha = 71.6040,\ \beta = 6.8407,\ \omega = 120 \tag{49}$$

Let us first numerically consider the solution function $\phi_1(t)$, generally given by (17) in Theorem 3.1. For α, β, and ω as given in (49), we obtain by (48) $\kappa_2 \approx 0.9297$, which together with $\kappa_0 \approx 0.3094$ (see (31)) gives for $\phi_1(t)$

$$\phi_1(t) = 2\cdot\arcsin\left(\mathrm{sn}\left(0.9297\cdot t - 0.3094\,|26.27\right)\right) - \frac{\pi}{8}$$

For u and v, we obtain by (35) and (47) $u \approx 92.59$ and $v \approx 8.691$; for the external moment function $f(t)$ this leads to

$$f(t) \approx 92.59\sin(\phi_1(t)) + 8.691\cos(\phi_1(t))$$

Concerning the ending time τ, we get for these values of the constants and the factor C_τ computed before, see (46) and (47),

$$\tau = C_\tau\cdot\frac{\omega}{\alpha\beta} \approx 0.6656$$

The functions $\phi_1(t)$ and $f(t)$ on the interval $[0,\tau]$ are plotted in the following figures (see Figs. 2 and 3):

As already mentioned before, these plots are totally unspectacular and naturally-looking; however, this is precisely the goal of our investigation, namely to show that by assuming a rather unnatural-seeming ansatz for the moment function, which is motivated by the desire to make the problem accessible to an analytic treatment by Jacobian elliptic functions, we can nevertheless produce a result which corresponds to what one naturally expects from the laboratory experiences.

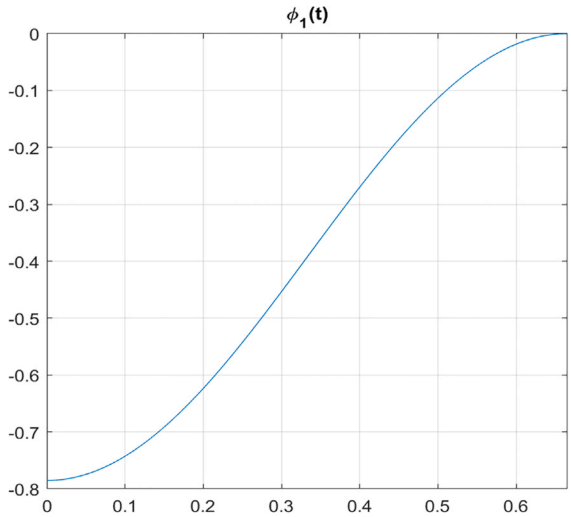

Fig. 2. $\phi_1(t)$.

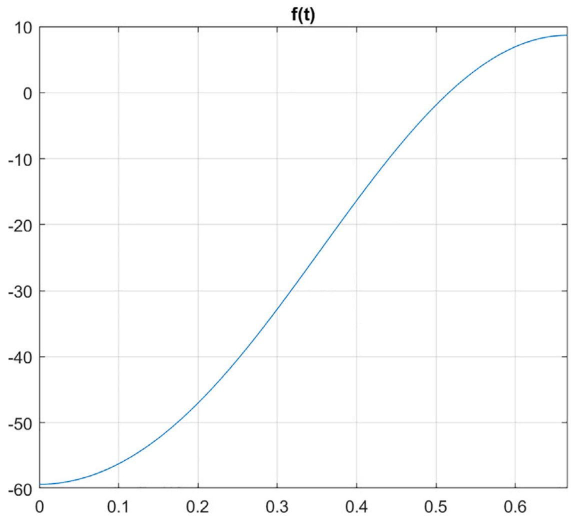

Fig. 3. $f(t)$.

6. Conclusion

We have shown that by assuming a special form of the external moment function $f(t)$ we can make the original, apparently ill-posed problem amenable to a treatment by Jacobian elliptic functions given by formula (17). The value of such an explicit solution lies, on the one hand, in its applicability for educational purposes and, on the other hand, in its surprisingly great similarity to the results of numerical simulations.

More precisely, by naively fitting the free parameters α, β, and τ to the experiments of the laboratory simulations, we can find a way to describe the system in a way that is both tractable by the analytical method of elliptic Jacobian functions and similar to what is observed in physical reality. To achieve better results, it should be possible to improve the approximation to reality by resorting to modern fitting methods.

As far as the pedagogical purpose is concerned, it is clear that the cubli system, from which our ODE system is derived, is a rather simple system intended to illustrate different models and methods of control theory to students, and not a one-to-one model for industrial applications. By showing that such a simple model can be made amenable to an analytical treatment by making a special ansatz to the external moment function, we hope that the results will show a way to make more complex control systems amenable to an analytical approach as well.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used DeepL Translator in order to obtain a basis for improving the writing style of some parts of the manuscript. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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Appendix. Closed solutions

As mentioned in the introduction, we consider as closed solution any solution of the initial value problem that can be expressed by the functions in the following list.

- *Incomplete elliptic integral of the first kind $F(\phi|m)$:*
  If $0 \leq m \leq \sin(\phi)^{-2}$, then Legendre's representation of $F(\phi|m)$ is

$$F(\phi|m) = \int_0^{\phi} \frac{\mathrm{d}\varphi}{\sqrt{1 - m \cdot \sin^2(\varphi)}}. \tag{50}$$

- *Incomplete elliptic integral of the second kind $E(\phi|m)$:*
  If $0 \leq m \leq \sin(\phi)^{-2}$, then Legendre's representation of $E(\phi|m)$ is

$$E(\phi|m) = \int_0^{\phi} \sqrt{1 - m \cdot \sin^2(\varphi)}\mathrm{d}\varphi.$$

- *Jacobian elliptic functions sn$(u|m)$, cn$(u|m)$, dn$(u|m)$:* They are defined as, using (50),

$$\mathrm{sn}(u|m) = \sin\left(F^{-1}(u|m)\right) \quad \text{and} \quad \mathrm{cn}(u|m) = \cos\left(F^{-1}(u|m)\right), \tag{51}$$

  as well as

$$\mathrm{dn}(u|m) = \mathrm{cn}\left(\sqrt{m} \cdot u \,\middle|\, \frac{1}{m}\right). \tag{52}$$

Remark. The modulus m of the elliptic integrals and the Jacobian elliptic functions can be extended to arbitrary values in $\mathbb{R}$.

We also need the following integral formulas:

$$\int \mathrm{dn}(u|m)\mathrm{d}u = \arcsin(\mathrm{sn}(u|m)) + C$$

$$\int \mathrm{cn}(u|m) \cdot \mathrm{sn}(u|m)\mathrm{d}u = -\frac{\mathrm{dn}(u|m)}{m} + C \tag{53}$$

$$\int \mathrm{sn}(u|m)^2\mathrm{d}u = -\frac{E(u|m)}{m} + \frac{u}{m} + C \tag{54}$$

For a substantial treatment of Jacobi's elliptic functions sn, cn, and dn, as well as the elliptic integrals $F(\phi|m)$ and $E(\phi|m)$ of the first and second kind see [50], Chapters 16 and 17.

References

[1] Gajamohan M, Merz M, Thommen I, D'Andrea R. The cubli: A cube that can jump up and balance. In: International conference on intelligent robots and systems. 2012, p. 3722–7.
[2] Gajamohan M, Muehlebach M, Widmer T, D'Andrea R. The cubli: A reaction wheel based 3D inverted pendulum. In: 2013 European control conference. 2013, p. 268–74.
[3] Lundberg KH, Barton TW. History of inverted-pendulum systems. IFAC Proc Vol 2010;42(24):131–5, 8th IFAC Symposium on Advances in Control Education.
[4] Boubaker O. The inverted pendulum: A fundamental benchmark in control theory and robotics. In: International conference on education and e-learning innovations. 2012, p. 1–6.
[5] Woude LVD, Franke N, Strien RV, Valkenstijn K, Klijn P, Langen H. The Kuutio: an autonomous balancing cube. In: 2017 International conference on research and education in mechatronics. 2017, p. 1–4.
[6] Hofer M, Muehlebach M, D'Andrea R. The one-wheel cubli: A 3D inverted pendulum that can balance with a single reaction wheel. Mechatronics 2023;91:102965.
[7] Muehlebach M, Mohanarajah G, D'Andrea R. Nonlinear analysis and control of a reaction wheel-based 3D inverted pendulum. In: 52nd IEEE conference on decision and control. 2013, p. 1283–8.
[8] Muehlebach M, D'Andrea R. Nonlinear analysis and control of a reaction- wheel-based 3-D inverted pendulum. IEEE Trans Control Syst Technol 2017;25(1):235–46.
[9] Åström KJ, Furuta K. Swinging up a pendulum by energy control. Automatica 2000;36:287–95.
[10] Graichen K, Zeitz M. A new approach to feedforward control design under output constraints applied to the side–stepping of the triple inverted pendulum. IFAC Proc Vol 2006;39(16):181–6, 4th IFAC Symposium on Mechatronic Systems.
[11] Singh R, Tayal VK, Singh HP. A review on cubli and non linear control strategy. In: 2016 IEEE 1st international conference on power electronics, intelligent control and energy systems. 2016, p. 1–5.
[12] Liao T-L, Chen S-J, Chiu C-C, Yan J-J. Nonlinear dynamics and control of a cube robot. Mathematics 2020;8(10).
[13] Bobrow F, Angelico BA, Martins FPR, da Silva PSP. The cubli: Modeling and nonlinear attitude control utilizing quaternions. IEEE Access 2021;9:122425–42.
[14] Li Z, Yang C, Fan L, editors. Advanced control of wheeled inverted pendulum systems. first ed.. Springer London; 2013.
[15] Boubaker O, editor. The inverted pendulum in control theory and robotics: From theory to new innovations. In: Control, robotics and sensors. Institution of Engineering and Technology; 2017.
[16] Harris J, Yazdani M. Development and control of a double-inverted pendulum. 2016.
[17] Łukowska A, Tomaszuk P, Ciezkowski M, Dzierżek K, Rećko M. Acceleration control approach of double inverted pendulum system. In: 2018 19th International Carpathian control conference. 2018, p. 122–7.
[18] Huang X, Wen F, Wei Z. Optimization of triple inverted pendulum control process based on motion vision.. J Image Video Proc 2018;73.

[19] Cui S, Liu Z. The optimal design of double inverted pendunum controller parameters based on ITAE. In: ICMLCA 2021; 2nd International conference on machine learning and computer application. 2021, p. 1–4.
[20] Yadav S, Sahebi AT, Juhász L. Adaptive model predictive controller for inverted pendulum. In: 2021 11th International conference on advanced computer information technologies. 2021, p. 208–13.
[21] Mahmoud MS, Saleh RAA, Ma'arif A. Stabilizing of inverted pendulum system using robust sliding mode control. Int J Robot Control Syst 2022;2(2):230–9.
[22] Gürsel Özmen N, Marul M. Stabilization and tracking control of an x-z type inverted pendulum system using lightning search algorithm tuned nonlinear PID controller. Robotica 2022;40(7):2428–48.
[23] Moatimid GM, El-Sayed AT, Salman HF. Dynamical analysis of an inverted pendulum with positive position feedback controller approximate uniform solution. Sci Rep 2023;13(1).
[24] Ardourel V, Jebeile J. On the presumed superiority of analytical solutions over numerical methods. Eur J Philos Sci 2017;7:201–20.
[25] Mendoza J, Muriel C. Exact solutions and Riccati-type first integrals. J Nonl Math Phys 2017;24(Supplement 1):75–89.
[26] Markakis MP. Exact solutions for certain nonlinear autonomous ordinary differential equations of the second order and families of two-dimensional autonomous systems. Int J Differ Equ Appl 2010;2010:13, Article ID 436860.
[27] Kudryashov NA. Nonlinear differential equations with exact solutions expressed via the weierstrass function. Z Naturforsch 2004;59a:443–54.
[28] Saxe AM, McClelland JL, Ganguli S. Exact solutions to the nonlinear dynamics of learning in deep linear neural networks. In: Proceedings of the international conference on learning represenatations. 2014.
[29] Bagchi S. Formulating analytical solution of network ODE systems based on input excitations. J Inf Process Syst 2018;14(2):455–68.
[30] Chen RTQ, Rubanova Y, Bettencourt J, Duvenaud DK. Neural ordinary differential equations. In: Bengio S, Wallach H, Larochelle H, Grauman K, Cesa-Bianchi N, Garnett R, editors. Advances in neural information processing systems, vol. 31. Curran Associates, Inc.; 2018.
[31] Dufera TT. Deep neural network for system of ordinary differential equations: Vectorized algorithm and simulation. Mach Learn Appl 2021;5:100058.
[32] Vitanov NK. Simple equations method (SEsM): An effective algorithm for obtaining exact solutions of nonlinear differential equations. Entropy 2022;24(11):1653.
[33] Ali HS, Alali E, Ebaid A, Alharbi FM. Analytic solution of a class of singular second-order boundary value problems with applications. Mathematics 2019;7(2).
[34] El-Dib YO, Elgazery NS, Alyousef HA. A heuristic approach to the prediction of a periodic solution for a damping nonlinear oscillator with the non-perturbative technique. Front Phys 2023;11.
[35] Turkyilmazoglu M. Approximate analytical solution of the nonlinear system of differential equations having asymptotically stable equilibrium. Filomat 2017;31(9):2633–41.
[36] Strogatz S. Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering. second ed.. CRC Press; 2015.
[37] Katok A, Hasselblatt B. Introduction to the modern theory of dynamical systems. Encyclopedia of mathematics and its applications, Cambridge University Press; 1995.
[38] Bates D, Watts D. Nonlinear regression analysis and its applications. New York: Wiley; 1988.
[39] Seber G, Wild C. Nonlinear regression. New York: Wiley; 2003.
[40] Ramsay J, Hooker G. Dynamic data analysis. New York: Springer; 2017.
[41] Prodanov D. Analytical solutions and parameter estimation of the SIR epidemic model. Math Anal Infecti Dis 2022.
[42] Lehtonen J. The Lambert W function in ecological and evolutionary models. Methods Ecol Evol 2016;7:1110–8.
[43] Berci M, Righi M. An enhanced analytical method for the subsonic indicial lift of two-dimensional aerofoils – with numerical cross-validation. Aerosp Sci Technol 2017;67:354–65.
[44] Berci M, Dimitriadis G. A combined multiple time scales and harmonic balance approach for the transient and steady-state response of nonlinear aeroelastic systems. J Fluids Struct 2018;80:132–44.
[45] Riso C, Cesnik CE, Epureanu BI. Output-based flutter bifurcation forecasting in nonlinear fluid-structural systems with multiple varying parameters. J Fluids Struct 2021;101:103201.
[46] Ronch AD, Ventura A, Righi M, Franciolini M, Berci M, Khralamov D. Extension of analytical indicial aerodynamics to generic trapezoidal wings in subsonic flow. Chin J Aeronaut 2018;31(4):617–31.
[47] Walter W. Gewönliche differentialgleichungen. Berlin Heidelberg New York: Springer-Verlag; 2000.
[48] Zwillinger D. Handbook of differential equations. Academic Press; 1997.
[49] Andrei D. Polyanin VFZ, editor. Handbook of ordinary differential equations. CRC Press; 2017.
[50] Abramowitz M, Stegun IA. Handbook of mathematical functions with formulas, graphs, and mathematical tables. NBS applied mathematics series, vol. 55, Wahington, DC: National Bureau of Standards; 1964.