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On the dimensions of automorphism groups of four-dimensional double loops

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The first systematic study of topological loops and double loops was presented by K.H. Hofmann in a series of papers at the end of the 50's. He considered double loops in the framework of topological algebra. For a comprehensive survey on the theory of loops and double loops the reader is referred to [7], in particular to chapters IX and XI. In a more geometrical setting, topological double loops appear together with coordinate domains of topological planes: every topological ternary field (T, τ) yields a double loop $(T, +, \circ)$ by setting $x + y = \tau(x, 1, y)$ and $x \circ y = \tau(x, y, 0)$. In general, the ternary operation τ cannot be reconstructed from the underlying double loop operations $+$ and $\circ$, see [1] and [29]. Little can be said about topological double loops in general, but locally compact connected double loops have a rich topological structure (see the appendix of this paper). By a deep result due to R. Löwen ([16] and [12]), their dimension can only be 1, 2, 4, 8, or, possibly, ∞. The classical examples of locally compact connected double loops are the real and complex numbers, the quaternions and the octonions.

In the work of H. Salzmann on compact connected projective planes, double loops and their automorphism groups are treated from a geometrical point of view. Salzmann's classification program of such planes is mainly based on the size of their automorphism groups. Since the automorphism group Δ of a compact connected projective plane $\mathscr{P}$ is locally compact with respect to the compact-open topology [23, §2], the (covering) dimension serves as a useful measure for the size of Δ. Any quadrangle Q in $\mathscr{P}$ leads to a locally compact connected ternary field T_Q and every (continuous) automorphism of T_Q corresponds to some (continuous) automorphism of $\mathscr{P}$ which fixes the quadrangle Q pointwise. In particular, $\Gamma = \operatorname{Aut} T_Q$ is locally compact. This result happens to be true also for the automorphism group of a double loop [3].

It is known that a one-dimensional double loop is rigid, i.e. it possesses no non-trivial automorphism (for a proof see [12, XI.9.1]). Similarly, the group of continuous automorphisms of a two-dimensional double loop contains at most two elements, see

[12, XI.9.3]. For eight-dimensional ternary fields, H. Salzmann proved in [24] that either the connected component of the automorphism group Γ is isomorphic to the compact simple exceptional Lie group G_2 as in the classical case, or Γ has dimension at most 13. It is unknown, however, whether or not this result can be extended to automorphism groups of eight-dimensional double loops.

In dimension four, the classical example is the field of quaternions, with automorphism group $SO_3\mathbb{R}$. Thus it is natural to expect that the automorphism group Γ of an arbitrary four-dimensional locally compact connected double loop has dimension at most 3. This is true, in fact, if Γ is compact [12, XI.9.7]. Without the last assumption, H. Salzmann proved $\dim \Gamma \leq 5$ for ternary fields; no bound was known so far for double loops.

In this paper, we show that $\dim \Gamma \leq 4$ holds in general for automorphism groups Γ of four-dimensional locally compact connected double loops. This result simplifies many arguments about eight-dimensional compact projective planes.

1 Definitions and notation

A *quasigroup* $\mathscr{L} = (L, \circ)$ is a set L together with a binary operation $\circ$ on L such that the equations $a \circ x = b$ and $y \circ a = b$ always have unique solutions x and y. These solutions are denoted by $x = a/b$ and $y = a\backslash b$. A *loop* $\mathscr{L} = (L, 1, \circ)$ is a quasigroup $(L, \circ)$ with a neutral element $1 \in L$, i.e. $1 \circ x = x = x \circ 1$ for every $x \in L$. A loop $\mathscr{L}$ is called *topological* iff L is a topological space which is neither discrete nor antidiscrete, such that the binary operations $\circ$, $/$, and $\backslash$ are continuous on $L \times L$. A *topological double loop* $\mathscr{D} = (D, 0, 1, +, \circ)$ is a topological space D such that $(D, 0, +)$ and $(D\backslash\{0\}, 1, \circ)$ are topological loops, $x \circ 0 = 0 = 0 \circ x$ holds for all $x \in D$, the multiplication $\circ$ is continuous on $D \times D$, and the mappings $x \mapsto x \circ a$ and $x \mapsto a \circ x$ are homeomorphisms of D for every $0 \neq a \in D$.

Throughout this paper, let $\mathscr{D}$ be a four-dimensional locally compact connected double loop and let Γ be a closed subgroup of the full automorphism group of $\mathscr{D}$. By [3], this group is a locally compact transformation group with respect to the compact-open topology. Thus the covering dimension as well as the small and the large inductive dimension coincide for Γ. Since we are only interested in the topological dimension of Γ, we may assume throughout the paper that the group Γ is connected (see the sum theorem in [21, 3.2.5]).

For a subset $M \subseteq \mathscr{D}$, the smallest closed sub-double-loop of $\mathscr{D}$ containing M is denoted by $\langle M \rangle$. If $M = \{0, 1\}$ we shall call $\mathscr{E} := \langle M \rangle$ the prime double loop of $\mathscr{D}$. For any subgroup Φ of Γ we denote by $\mathscr{F}_\Phi$ the set of all those elements of $\mathscr{D}$ that are fixed by every automorphism $\phi \in \Phi$. Clearly, $\mathscr{F}_\Phi$ is a closed sub-double-loop of $\mathscr{D}$. The one-point compactification $\mathscr{D} \cup \{\infty\}$ of $\mathscr{D}$ is denoted by $\widehat{\mathscr{D}}$. All homology groups are assumed to be singular homology groups, whereas the cohomology groups are used in the sense of Alexander-Spanier-Čech (see e.g. [28, Chap.6, Sect. 4] or [17, Chap. IX, §6]). Reduced (co-)homology groups are written with a tilde on top.

2 Dimensions of automorphism groups

For compact groups Γ the sharp bound $\dim \Gamma \leq 3$ was proved by H. Salzmann in [12, XI.9.7]. For non-compact groups Γ the inequality $\dim \Gamma \leq 4$ can be verified at once if we assume that the double loop $\mathcal{F}_\Gamma$ of fixed elements is connected.

Lemma 2.1 *If the double loop $\mathcal{F}_\Gamma$ of fixed elements is connected, then we have* $\dim \Gamma \leq 4$.

Proof . We may assume that $\mathcal{F}_\Gamma \neq \mathcal{L}$. Since $\mathcal{F}_\Gamma$ is connected, we thus have $1 \leq \dim \mathcal{F}_\Gamma \leq 2$ by (3.1) and (3.3). Hence there exist no zero-dimensional double loops in $\mathcal{L}$, see [12, XI.9.2] and the proof of [12, XI.9.1]. In particular, the prime double loop $\mathcal{E}$ is connected. Consequently we have $\dim \langle c \rangle \geq 2$ for any element $c \in \mathcal{L} \setminus \mathcal{F}_\Gamma$ since $\mathcal{E} < \langle c \rangle$. If $\dim \langle c \rangle = 4$ we have $\Gamma_c = \mathbb{1}$ and thus $\dim \Gamma \leq \dim \mathcal{L} = 4$ by [13]. Thus by (3.3), we may assume that $\dim \langle c \rangle = 2$ for every $c \in \mathcal{L} \setminus \mathcal{F}_\Gamma$ and we may choose elements $c, d \in \mathcal{L} \setminus \mathcal{F}_\Gamma$ such that $\langle c, d \rangle = \mathcal{L}$. By [25, §2, (6)] the orbits c^Γ and d^Γ are at most two-dimensional. Note that [25, §2, (6)] is formulated for four-dimensional ternary fields rather than for double loops, but the proof only uses the fact that a two-dimensional ternary subfield has at most two continuous automorphisms. This is also true for two-dimensional double loops (see [12, XI.9.3]). Applying [13] we therefore obtain the desired inequality

$$\dim \Gamma = \dim \Gamma_c + \dim c^\Gamma = \dim \Gamma_{c,d} + \dim d^{\Gamma_c} + \dim c^\Gamma \leq 0 + 2 + 2 = 4.$$

Combining this result with Lemma (3.5) of the appendix and [12, XI.9.2] we get the following very useful corollary.

Corollary 2.2 *If Γ contains a non-trivial element of finite order, then the dimension of the group Γ is at most four.*

In the next step we shall study the dimension of Γ in the case where Γ is semi-simple.

Corollary 2.3 *A non-trivial quasi-simple automorphism group Γ is three-dimensional.*

Proof . The quotient group $\Gamma^* := \Gamma / \mathrm{Z}(\Gamma)$ is a simple Lie group, where the center $\mathrm{Z}(\Gamma)$ is a zero-dimensional group. Assume that $\dim \Gamma > 3$. Then, by the classification of quasi-simple Lie groups we have $\dim \Gamma = \dim \Gamma^* \geq 6$ and there is a non-trivial compact connected subgroup Φ^* of Γ^* which is covered by a compact connected Lie subgroup Φ of Γ. The Lie group Φ contains a torus subgroup and thus has non-trivial elements of finite order. Hence the assertion follows by Corollary (2.2).

In order to show that a semi-simple automorphism group Γ of $\mathcal{L}$ is in fact quasi-simple, we first need information about subgroups of Γ centralizing each other.

Lemma 2.4 *If Φ, Ψ are two non-trivial connected subgroups of Γ centralizing each other, then the inequality* $\dim \Phi \leq 4$ *holds. Furthermore, if* $\dim \Phi = 4$, *then we have*

$\dim \mathscr{F}_\Phi = 2$, *$\Phi$ is a Lie group, $\mathscr{D}$ is homeomorphic to* $\mathbb{R}^4$, *and Φ is homeomorphic to* $\mathbb{R}^3 \times \mathbb{T}$, *where $\mathbb{T}$ denotes the circle group.*

Proof . Since the connected group Φ leaves $\mathscr{F}_\Psi$ invariant and the dimension of $\mathscr{F}_\Psi$ is at most 2, the group Φ must fix $\mathscr{F}_\Psi$ elementwise. Interchanging the roles of Φ and Ψ, it follows that $\mathscr{F} := \mathscr{F}_\Phi = \mathscr{F}_\Psi$. For any $c \in \mathscr{D}\backslash\mathscr{F}$ we have $\langle c^\Psi \rangle = \mathscr{D}$ by [12, XI.9.1, XI.9.3], since Ψ is a non-trivial connected group. Because Φ centralizes Ψ, we obtain $\Phi_c = \mathbb{1}$, i.e. the group Φ acts freely on $\mathscr{D}\backslash\mathscr{F}$. Hence $\dim \Phi \leq \dim \mathscr{D} = 4$ holds. Now let $\dim \Phi = 4$. Then Φ acts transitively on $\mathscr{D}\backslash\mathscr{F}$ by Corollary (3.6) of the appendix. Hence Φ is a Lie group and $\mathscr{D}$ is a topological manifold by [19, (6.3)]. Thus $\mathscr{D}$ is homeomorphic to $\mathbb{R}^4$ by (3.2). Having a non-trivial maximal compact subgroup by [25, §2 (11)], the group Φ contains a non-trivial element α of odd order. Since Φ acts freely on $\mathscr{D}\backslash\mathscr{F}$, we have $\dim \mathscr{F} = \dim \mathscr{F}_\alpha = 2$ by (3.5). Next we shall determine the topological structure of the group Φ by using Alexander duality. Note first that by [12, XI.9.6] the group Φ cannot contain a two-dimensional torus group. Being a four-dimensional non-compact Lie group with a non-trivial maximal compact subgroup, the group Φ is thus homeomorphic to one of the following spaces: $\mathbb{R}^3 \times \mathbb{T}$, $\mathbb{R} \times \mathrm{P}_3\mathbb{R}$, or $\mathbb{R} \times \mathbb{S}_3$. Since the one-point compactification $\widehat{\mathscr{F}_\Phi}$ of $\mathscr{F}_\Phi$ is homeomorphic to the 2-sphere by [12, XI.8.2.c] we obtain from Alexander duality the relation

$$\widetilde{\mathbf{H}}_k(\Phi) \cong \widetilde{\mathbf{H}}_k(\mathscr{D}\backslash\mathscr{F}_\Phi) \cong \widetilde{\mathbf{H}}_k(\widehat{\mathscr{D}}\backslash\widehat{\mathscr{F}_\Phi}) \cong \widetilde{\mathbf{H}}^{3-k}(\widehat{\mathscr{F}_\Phi}) \cong \widetilde{\mathbf{H}}^{3-k}(\mathbb{S}_2)$$

for $0 \leq k \leq 3$. Now the relations $\mathbf{H}^2(\mathbb{S}_2) \cong \mathbb{Z}$ and $\widetilde{\mathbf{H}}^{3-k}(\mathbb{S}_2) = 0$ for $k \neq 1$ exclude all but one of the spaces mentioned above. The remaining space is just $\mathbb{R}^3 \times \mathbb{T}$, which finishes the proof.

Theorem 2.5 *A non-trivial semi-simple automorphism group Γ is a three-dimensional quasi-simple group.*

Proof. Since Γ is semi-simple, the center Z of Γ is zero-dimensional and the quotient group $\Gamma/_\mathrm{Z}$ can be written as a product

$$\Gamma/_\mathrm{Z} = \prod_k \Psi_k\mathrm{Z}/_\mathrm{Z},$$

where the factors $\Psi_k^* := \Psi_k\mathrm{Z}/_\mathrm{Z}$ are connected simple Lie groups centralizing each other. Note that the groups Ψ_k are coverings of the groups Ψ_k^*. By Corollary (2.3), we may assume that we have at least two distinct non-trivial factors Ψ_1 and Ψ_2. If both of the groups Ψ_1^* and Ψ_2^* are compact, their coverings Ψ_1 and Ψ_2 are compact quasi-simple Lie groups. This would imply that Γ contains a two-dimensional torus, which is impossible by [12, XI.9.7]. Hence we may assume that Ψ_1^* contains a closed one-parameter subgroup $\mathrm{R}^* \cong \mathbb{R}$. This subgroup is trivially covered in Ψ_1 by a closed one-parameter subgroup $\mathrm{R} \cong \mathbb{R}$ which is centralized by Ψ_2. Applying Lemma (2.4) with $\Phi = \mathrm{R}\overline{\Psi_2}$ and $\Psi = \mathrm{R}$ to the product $\mathrm{R}\overline{\Psi_2}$, we infer that this group is a Lie group homeomorphic to $\mathbb{R}^3 \times \mathbb{T}$. In particular, the group Γ contains a non-trivial element of finite order and by Corollary (2.2) we conclude that $\dim \Gamma \leq 4$. On the other hand, the group Γ contains the two quasi-simple factors Ψ_1 and Ψ_2 and thus we have $\dim \Gamma \geq 6$, which is a contradiction. Hence the automorphism group Γ must be quasi-simple and the theorem follows by Corollary (2.3).

The following lemma will play the key role in the proof of the inequality $\dim \Gamma \leq 4$. As an immediate consequence of this lemma we have $\dim \Gamma \leq 5$.

Lemma 2.6 *If Γ is a non-semisimple automorphism group of $\mathscr{D}$ with* $\dim \Gamma_c \geq 2$ *for some $c \in \mathscr{D}\backslash\mathscr{F}_\Gamma$, then* $\dim \Gamma \leq 4$.

Proof. Being non-semisimple, the group Γ contains a minimal connected commutative normal subgroup Ξ. If the normal subgroup Ξ is compact, then Ξ lies in the center of Γ, since the connected group Γ induces isomorphic automorphism groups on both Ξ and the discrete dual Ξ^* (see e.g. [14, (26.20) Theorem.]). This implies $\dim \Gamma \leq 4$ by (2.4). Hence we may assume Ξ to be non-compact. Minimality of Ξ now implies that $\Xi \cong \mathbb{R}^n$. Let Z be the center of Γ and set $\Theta := C_\Gamma \Xi$. Since Ξ is a normal subgroup in Γ, the fix-double-loop $\mathscr{F}_\Xi$ of Ξ is Γ-invariant. This implies that $\mathscr{F}_\Xi = \mathscr{F}_\Gamma$, for otherwise Γ would act non-trivially on $\mathscr{F}_\Xi$ and therefore $\dim \mathscr{F}_\Xi \geq 4$ would hold by [12, XI.9.1, XI.9.3]. Hence $\mathscr{F}_\Xi = \mathscr{D}$, which is impossible. In particular, we have $c^\Xi \neq c$ and as before we infer that $\langle c^\Xi \rangle = \mathscr{D}$. Thus the stabilizer Θ_c is trivial. Now let $\Pi < \Xi$ be a minimal Γ_c^1–invariant subspace, where Γ_c^1 denotes the connected component of the stabilizer Γ_c containing the identity. Arguing as before, we get $\langle c^\Pi \rangle = \mathscr{D}$ and therefore Γ_c^1 must act faithfully and irreducibly on Π. Furthermore, the stabilizer Γ_c is a Lie group by the argument of [24, (3.3)] and Γ_c^1 is a reductive group by [11, §19.14, §19.17] or [4, Chap. I, 6.4., Prop. 5, p.56]. In particular, there is a linear semi-simple Lie group Σ (possibly $\Sigma = \mathbb{1}$) such that $\Sigma \leq \Gamma_c^1 \leq \Sigma\mathbb{C}^\times$. Since we have $\dim \Gamma_c^1 \geq 2$ by hypothesis, the group Γ_c^1 must contain a torus subgroup. Thus the assertion of the lemma follows from Corollary (2.2).

We are now able to prove the main result of this paper.

Theorem 2.7 *The full automorphism group Γ of a four-dimensional locally compact double loop $\mathscr{D}$ is of dimension at most four.*

Proof. In view of Theorem (2.5) we may assume Γ to be a non-semisimple group. Then Lemma (2.6) implies that $\dim \Gamma \leq 5$, so we may assume in the sequel that $\dim \Gamma = 5$. In this case all orbits c^Γ for $c \in \mathscr{D}\backslash\mathscr{F}_\Gamma$ are four-dimensional by Lemma (2.6). Hence, these orbits are open subsets of $\mathscr{D}$ and the group Γ acts transitively on the complement $\mathscr{D}\backslash\mathscr{F}_\Gamma$ by Lemma (3.4), since $\mathscr{D}$ is a Cantor manifold by [26, Thm.A]. Furthermore, the group Γ is a Lie group und $\mathscr{D}$ is homeomorphic to $\mathbb{R}^4$, compare the proof of (2.4). By Corollary (2.2) the group Γ cannot contain a non-trivial element of finite order and thus the maximal compact subgroup of Γ is trivial. By the Malcev-Iwasawa theorem (see [15, Thm. 13, p. 549]) this implies that Γ is homeomorphic to $\mathbb{R}^5$. Finally, we may assume by Proposition (2.1) that the double loop $\mathscr{F}_\Gamma$ of fixed elements is zero-dimensional. Now, by (3.1) we know that $\mathscr{D}$ is locally connected and we may apply Lemma (3.4) to obtain that $\Gamma/\Gamma_c \approx c^\Gamma = \mathscr{D}\backslash\mathscr{F}_\Gamma$. To obtain a contradiction in this situation, we shall show that the homotopy groups of Γ/Γ_c and of $\mathscr{D}\backslash\mathscr{F}_\Gamma$ are different.

We first note that the reduced cohomology group $\widetilde{\mathbf{H}}^0(\widehat{\mathscr{F}_\Gamma})$ does not vanish, because $\widehat{\mathscr{F}_\Gamma}$ is totally disconnected and contains at least three elements, namely the elements 0, 1, and ∞. Moreover, the higher cohomology groups $\mathbf{H}^k(\widehat{\mathscr{F}_\Gamma})$ do vanish for $k > 0$, since $\dim \mathscr{F}_\Gamma = 0$. Applying Alexander duality to $\widehat{\mathscr{D}}$ and $\widehat{\mathscr{F}_\Gamma}$, we obtain for $0 \leq k \leq 3$

the isomorphism

$$\widetilde{\mathbf{H}}_k\left(\Gamma/_{\Gamma_c}\right) \cong \widetilde{\mathbf{H}}_k\left(c^\Gamma\right) \cong \widetilde{\mathbf{H}}_k\left(\mathscr{L}\setminus\mathscr{F}_\Gamma\right) \cong \widetilde{\mathbf{H}}^{3-k}\left(\widehat{\mathscr{F}_\Gamma}\right).$$

Note that for formulating Alexander duality for non-manifolds $\widehat{\mathscr{F}_\Gamma}$ we have to take Alexander-Spanier cohomology groups, see e.g. [17, Th. 6.6, p.222–223] or [8, VIII, 8.15]. Hence we have $\widetilde{\mathbf{H}}_k\left(\Gamma/_{\Gamma_c}\right) = 0$ if $0 \leq k \leq 2$ and $\mathbf{H}_3\left(\Gamma/_{\Gamma_c}\right) \neq 0$. Because the complement $\mathscr{L}\setminus\mathscr{F}_\Gamma$ is simply connected by [2], we can apply the Hurewicz isomorphy (see e.g. [28, Chap.7, Sect.5, Thm.4, p. 397]) to get the relations $\pi_k(\Gamma/_{\Gamma_c}) = \mathbb{1}$ if $0 \leq k \leq 2$ and $\pi_3(\Gamma/_{\Gamma_c}) \neq \mathbb{1}$. On the other hand, the short exact sequence

$$0 \longrightarrow \Gamma_c \longrightarrow \Gamma \longrightarrow \Gamma/_{\Gamma_c} \longrightarrow 0$$

induces a long exact sequence in homotopy

$$\ldots \to \pi_{n+1}(\Gamma/_{\Gamma_c}) \to \pi_n(\Gamma_c) \to \pi_n(\Gamma) \to \pi_n(\Gamma/_{\Gamma_c}) \to \pi_{n-1}(\Gamma_c) \to \ldots.$$

Because the group Γ is homeomorphic to $\mathbb{R}^5$ and the stabilizer Γ_c^1 is homeomorphic to $\mathbb{R}$, all homotopy groups of Γ and Γ_c vanish for $n \geq 1$. In particular, the exactness of the homotopy sequence implies that $\pi_3(\Gamma/_{\Gamma_c}) \cong \pi_3(\Gamma_c) = \mathbb{1}$, which contradicts the previous result obtained from Alexander duality. Hence the assumption $\dim \Gamma = 5$ is contradictory, and therefore the inequality $\dim \Gamma \leq 4$ must hold.

3 Appendix

The results in this section hold for locally compact connected double loops of arbitrary dimension.

Theorem 3.1 *A locally compact double loop $\mathscr{L}$ is a doubly homogeneous separable complete metric space which is σ-compact but not compact and has a countable basis. Moreover, $\mathscr{L}$ is either connected or totally disconnected. If $\mathscr{L}$ is totally disconnected, then the one-point compactification $\widehat{\mathscr{L}}$ of $\mathscr{L}$ is homeomorphic to the Cantor set. If $\mathscr{L}$ is connected, then it is locally connected and path connected as well as locally and globally contractible. Furthermore, in this case we have $\dim \mathscr{L} > 0$.*

For a proof see XI.1.2 to XI.1.5 and XI.8.1, XI.8.3, XI.8.4 of [12].

Theorem 3.2 *Let $\mathscr{L}$ be a locally compact connected double loop of finite covering dimension n. Then $\mathscr{L}$ is a Cantor manifold and an ANR which has domain invariance. Moreover, any closed n-dimensional subset of $\mathscr{L}$ contains inner points. If $\mathscr{L}$ is a topological manifold, then it is homeomorphic to $\mathbb{R}^n$.*

The proofs of these statements can be found in [16] and [26], cp. also [12, XI.8]. For the last assertion see [22, 7.12] and [16, 5.2].

Theorem 3.3 *Let $\mathscr{L}$ be a locally compact connected double loop of finite covering dimension n. Then $\widehat{\mathscr{L}}$ is an n-dimensional homology and cohomology manifold over* *an arbitrary principal ideal domain L, and n is also the cohomological dimension of $\mathscr{S}$. Moreover, $\widehat{\mathscr{S}}$ is a homotopy n-sphere and we have $n = 2^\ell$ with $\ell \leq 3$.*

Proof . In [16, Thm. 2a] it is proved that a locally compact connected ternary field is an n-dimensional homology manifold over $\mathbb{Z}$. Since the proof of this fact only uses the double loop structure of a ternary field this result applies to double loops as well (cp. [12, p.334]). By definition, a locally contractible space is clc_L^∞ for every principal ideal domain L (see [5, Def. 16.1, p.76]). Moreover, the stalk $\mathscr{H}_n(\mathscr{S}; L)$ is constant over $\mathscr{S}$ by [16, Lemma 6.1 and p. 113], which implies that $\mathscr{S}$ is an n-dimensional homology manifold over an arbitrary principal ideal domain L (see [5, 15.2, p.240] and [6, p. 469]). Finally, the notions of a homology manifold and a cohomology manifold coincide for locally contractible spaces, see [16, p. 114]. The last assertion is proved in [16] and [12, XI.8.5].

Lemma 3.4 *Let Γ be a locally compact Lindelöf group acting on a separable metric space M which is locally compact, locally homogeneous, and locally contractible. If $\dim a^\Gamma = \dim M$ for some $a \in M$, then the orbit a^Γ is open in M and a^Γ is homeomorphic to the homogeneous space Γ / Γ_a. In particular this holds for a locally compact connected automorphism group of a locally compact connected double loop.*

Proof . Let $\Omega \subset \Gamma$ be an arbitrary compact neighborhood. Because Γ is a Lindelöf group, there exist elements $\gamma_n \in \Gamma$ $(n \in \mathbb{N})$ such that

$$a^\Gamma = \bigcup_{\gamma \in \Gamma} a^{\Omega\gamma} = \bigcup_{n \in \mathbb{N}} a^{\Omega\gamma_n}.$$

By the sum theorem (see [21, 3.2.5]) there is thus an integer $m \in \mathbb{N}$ satisfying $\dim a^\Gamma = \dim a^{\Omega\gamma_m}$. Since γ_m is a homeomorphism, we conclude that $\dim a^\Omega = \dim a^{\Omega\gamma_m} = \dim a^\Gamma = \dim M$. The compactness of Ω implies the compactness of a^Ω and by [26, Thm.C] the orbit a^Ω contains inner points. In particular, the orbit a^Γ contains inner points, and since Γ acts transitively on a^Γ, this orbit is an open set. Thus the set a^Γ is locally compact and [10] implies that $\gamma \mapsto a^\gamma : \Gamma \to a^\Gamma$ is an open map. Since this map is also closed (note that Γ is locally compact), we finally have $a^\Gamma \approx \Gamma / \Gamma_a$.

Lemma 3.5 *If $\mathbb{1} \neq \gamma \in \Gamma$ is an element of finite order, then $\dim \mathscr{F}_\gamma \geq 1$. Moreover, if γ is not an involution, then $\dim \mathscr{F}_\gamma$ is even.*

Proof . Because $\widehat{\mathscr{S}}$ is a compact n-dimensional cohomology manifold over $\mathbb{Z}_p$ as well as an n-dimensional cohomology sphere for every prime number p by Theorem (3.3), we may apply [18, Thm.C, p. 463] to the group $\langle\gamma\rangle$ acting on $\widehat{\mathscr{S}}$ and obtain that $\mathscr{F}_\gamma$ is a (co-)homology-k-sphere. Note that by setting $\infty^\gamma = \infty$, the automorphism γ becomes a homeomorphism on $\widehat{\mathscr{S}}$. Since a homology-0-sphere only consists of two points and since γ fixes the three distinct points 0, 1, and ∞, we conclude that $k > 0$. By [18], this implies the connectedness of $\mathscr{F}_\gamma$. For the second part of the lemma see [9, Thm. 5.2]; compare also [27, p. 404].

Lemma 3.6 *If* $\dim c^\Gamma = \dim \mathscr{D}$ *for all* $c \in \mathscr{D}\backslash\mathscr{F}_\Gamma$, *then* $c^\Gamma = \mathscr{D}\backslash\mathscr{F}_\Gamma$, *and* Γ *is not compact.*

Proof. By [26, Thm.A] we know that $\mathscr{D}$ is a Cantor manifold. Since $\dim \mathscr{D} = 2^m \geq 4$ by [12, XI.8.5], we have $\dim \mathscr{F}_\Gamma \leq \dim \mathscr{D} - 2$ and thus the complement $\mathscr{D}\backslash\mathscr{F}_\Gamma$ ist connected. By (3.4) and the hypothesis, the complement $\mathscr{D}\backslash\mathscr{F}_\Gamma$ is a union of open Γ-orbits. Thus $c^\Gamma = \mathscr{D}\backslash\mathscr{F}_\Gamma$ since $\mathscr{D}\backslash\mathscr{F}_\Gamma$ is connected. In particular, the orbit c^Γ is locally compact and by Lemma (3.4) we have $c^\Gamma \approx \Gamma/\Gamma_c$. If we assume that the group Γ is compact, then the orbit $c^\Gamma = \mathscr{D}\backslash\mathscr{F}_\Gamma$ would be compact too. But this would imply that the set $\mathscr{D}\backslash\mathscr{F}_\Gamma$ is open and closed in $\mathscr{D}$, contradicting the connectedness of $\mathscr{D}$.

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