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On the Dimensions of Automorphism Groups of Eight-Dimensional Double Loops

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Abstract. Let $\mathscr{D}$ be an eight-dimensional, locally compact, connected double loop. It is proved that the dimension of the automorphism group Aut $\mathscr{D}$ with respect to the compact-open topology is at most 16.

Throughout this paper, let $\mathscr{D}$ denote a locally compact, connected double loop and let Γ be a closed subgroup of the full automorphism group of $\mathscr{D}$, which is taken with the compact-open topology. By [14] and [9, XI.8.5], the (covering) dimension of $\mathscr{D}$ is 1, 2, 4, 8, or, possibly, ∞. Excepting the appendix and Lemma (1.1), the dimension of $\mathscr{D}$ is always assumed to be eight. By [2], the group Γ is a locally compact transformation group of $\mathscr{D}$ and so the covering dimension coincides with both inductive dimensions (see [18, Thm. 2.1]). The same is true for the double loop $\mathscr{D}$ (see Lemma (3.1) of the appendix). In the case where $\mathscr{D}$ is an eight-dimensional ternary field, H. Salzmann has shown in [21] that either the connected component of Γ is isomorphic to the compact exceptional simple Lie group G_2, or $\dim \Gamma < 14$. This result depends on a compactness criterion for the group Γ (see [21, (2.2)]). For double loops such a criterion is not yet known.

For a subset $M \subseteq \mathscr{D}$, the smallest closed sub-double-loop of $\mathscr{D}$ containing M is denoted by $\langle M \rangle$. The double loop $\langle M \rangle$ is said to be generated by M. If $\langle M \rangle \neq \mathscr{D}$, then $\dim \langle M \rangle \in \{0, 1, 2, 4\}$ by the above result. Note that no example of a locally compact, connected double loop containing a 0-dimensional double loop is known. We shall call $\mathscr{E} := \langle 1 \rangle$ the prime double loop of $\mathscr{D}$. For any subgroup Φ of Γ, we denote by $\mathscr{F}_\Phi$ the set of all those elements of $\mathscr{D}$ that are fixed by every automorphism $\varphi \in \Phi$. Clearly, $\mathscr{F}_\Phi$ is a closed sub-double-loop of $\mathscr{D}$. If Φ leaves a sub-double-loop $\mathscr{H}$ of $\mathscr{D}$ invariant, it induces on

$\mathscr{H}$ an automorphism group $\Phi|_{\mathscr{H}}$, see also (3.2) of the appendix. We shall write $\Gamma_{[\mathscr{H}]}$ for the (closed) subgroup of Γ that fixes $\mathscr{H}$ pointwise. The one-point compactification $\mathscr{D} \cup \{\infty\}$ of $\mathscr{D}$ is denoted by $\hat{\mathscr{D}}$; it is homotopy equivalent to $\mathbb{S}_8$, see [14] and [9, XI.8.5]. All occurring homology groups are assumed to be singular homology groups with coefficient domain $\mathbb{Z}$, whereas the cohomology groups are used in the sense of Alexander–Spanier–Čech (see e.g. [24, Chap. 6, Sect. 4] or [15, Chapt. IX, §6]). Reduced (co-)homology groups are written with a tilde on top. When speaking about dimension in general, we always mean the covering dimension dim. Unless stated otherwise, we may assume by the sum theorem (see [19, §3, 2.5]) that the group Γ is connected, since we are only interested in the topological dimension of Γ. The center of Γ is denoted by $\mathsf{Z}(\Gamma)$. We use the symbol $\mathbb{T}$ for the circle group.

1. The Dimensions of Γ-Orbits

To obtain upper bounds for $\dim \Gamma$, we first have to establish non-trivial upper bounds for the dimensions of Γ-orbits. The following lemma generalizes a result of H. Salzmann [22].

1.1. Lemma. *Let* $\dim \mathscr{D} \geqslant 2$. *For any element* $c \in \mathscr{D}$ *which generates a two-dimensional sub-double-loop of* $\mathscr{D}$, *the inequality*

$$\dim c^{\Gamma} \leqslant \dim \mathscr{D} - 2$$

holds.

Proof. Let $\mathscr{C} := \langle c \rangle$ be a two-dimensional sub-double-loop. We may assume that $c \notin \mathscr{F}_{\Gamma}$, else the inequality stated in the lemma holds trivially. Then $\mathscr{F} := \mathscr{F}_{\Gamma} \cap \mathscr{C} < \mathscr{C}$ and therefore $\dim \mathscr{F} = 1$ by [9, XI.9.2]. Moreover, the sub-double-loop $\mathscr{C}$ is generated by any element of the set $\mathscr{C} \backslash \mathscr{F}$. Consider the continuous map

$$\eta : \mathscr{C} \backslash \mathscr{F} \times \Gamma \to \mathscr{D} : (x, \gamma) \mapsto x^{\gamma^{-1}}.$$

Because for any element $x \in \mathscr{D}$ the preimage $\eta^{\leftarrow}(x)$ is a closed subset of $\mathscr{C} \backslash \mathscr{F} \times \Gamma$, the monotony theorem (see [19, §6, 6.2]) implies that

$$\dim \eta^{\leftarrow}(x) \leqslant \dim (\mathscr{C} \backslash \mathscr{F} \times \Gamma).$$

Since $\dim \langle c \rangle = 2$ and $\dim \langle M \rangle \in \{2, 4, 8\}$ if $c \in M$, we can find two elements $x, y \in \mathscr{D}$ such that $\mathscr{D} = \langle c, x, y \rangle$. Thus, the stabilizer $\Gamma_{c,x,y}$ is trivial and we conclude that $\dim \Gamma \leqslant 3 \dim \mathscr{D} < \infty$ by repeated application of the dimension formula [10]. Hence, we have $\dim(\mathscr{C} \backslash \mathscr{F} \times$

$\times \Gamma) < \infty$. In particular, there is some element $d \in \mathscr{D}$ with $\dim \eta^{\leftarrow}(d) = \dim \eta := \sup_{x \in \mathscr{D}} \dim \eta^{\leftarrow}(x)$. Select arbitrary compact neighborhoods $U \subseteq \mathscr{C} \backslash \mathscr{F}$ and $\Omega = \Omega^{-1} \subseteq \Gamma$, and let η^* be the restriction of η to $U \times \Omega$. Since $U \times \Omega$ is compact, the map $\eta^*: U \times \Omega \to U^{\Omega}$ is a closed surjection, and thus we obtain from [19, §9, 2.6] the inequality

$$\dim (U \times \Omega) \leqslant \dim U^{\Omega} + \dim \eta^*.$$

Using the sum theorem [19, §3, 2.5] and Lemma (3.1) of the appendix, this yields

$$\dim \mathscr{C} + \dim \Gamma = \dim U + \dim \Omega = \dim (U \times \Omega) \leqslant \dim \mathscr{D} + \dim \eta^*.$$

In particular, we have

$$\dim \Gamma - \dim \eta^* \leqslant \dim \mathscr{D} - 2.$$

Thus we have to prove the inequality $\dim \eta^* \leqslant \dim \Gamma_c$, since $\dim \Gamma = \dim \Gamma_c + \dim c^{\Gamma}$ holds by [10]. Moreover, it is sufficient to verify that

$$\dim \eta^{\leftarrow}(d) \leqslant \dim \Gamma_c,$$

because we have $\dim \eta^* \leqslant \dim \eta = \dim \eta^{\leftarrow}(d)$. Fix an element $b \in d^{\Gamma} \cap \mathscr{C} \backslash \mathscr{F}$. Then the sub-double-loop $\mathscr{C}$ is invariant under an automorphism $\gamma \in \Gamma$ if and only if the element b^{γ} lies in $\mathscr{C}$, since $\mathscr{C}$ is generated by any element $x \in \mathscr{C} \backslash \mathscr{F}$. Because a connected two-dimensional double loop has at most two continuous automorphisms (see [9, XI.9.3]), the set $\mathscr{C} \cap b^{\Gamma}$ thus contains at most two elements b and b'. Select $\beta \in \Gamma$ with $b^{\beta} = b'$ and choose an automorphism $\delta \in \Gamma$ with $b^{\delta^{-1}} = d$. Setting

$$A_1 := \{(b, \gamma) \in \{b\} \times \Gamma \mid b = b^{\delta^{-1}\gamma}\}$$

and

$$A_2 := \{(b', \gamma) \in \{b'\} \times \Gamma \mid b' = b^{\delta^{-1}\gamma}\}$$

we obtain

$$\eta^{\leftarrow}(d) = A_1 \cup A_2.$$

The sets A_1 and A_2 have identical dimensions, because they are homeomorphic via the map $(b, \gamma) \mapsto (b^{\beta}, \gamma^{\beta})$. Thus the sum theorem yields that $\dim \eta^{\leftarrow}(d) = \dim A_1$. The set A_1 is homeomorphic to $\{\gamma \in \Gamma \mid b = b^{\delta^{-1}\gamma}\} = \{\alpha \in \delta^{-1}\Gamma \mid b = b^{\alpha}\} \approx \Gamma_b$. Finally, we have $\Gamma_b = \Gamma_c$, because $\mathscr{C} = \langle b \rangle = \langle c \rangle$, and the lemma is proved.

The next lemma studies the double loop $\mathscr{F}_\Gamma$ of fixed elements if Γ is a finite (non-connected) elementary abelian group or a torus group of rank two.

1.2. Lemma. *Let* $\Gamma \cong \mathbb{Z}_p^2$ *for some prime number p. Then* $\mathscr{F}_\Gamma$ *is one- or two-dimensional. If* $p = 2$ *and* $\dim \mathscr{F}_\Gamma = 2$, *then* $\dim \mathscr{F}_\gamma = 4$ *for every* $\gamma \in \Gamma \setminus \{\mathbb{1}\}$. *If* $\Gamma \cong \mathbb{T}^2$, *then* $\mathscr{F}_\Gamma$ *is two-dimensional.*

Proof. Let $\Gamma \cong \mathbb{Z}_p^2$. Then the fix-double-loop $\mathscr{F}_\Gamma$ is connected by [3, (3.3)]. If $\mathscr{F}_\Gamma$ would be four-dimensional, all elements of Γ would fix $\mathscr{F}_\Gamma$ pointwise which is impossible by [6, p. 262]. Thus $\mathscr{F}_\Gamma$ is either one- or two-dimensional. Now let $p = 2$ and $\dim \mathscr{F}_\Gamma = 2$. Let a, b, and c denote the dimensions of the fix-double-loops of the three elements in $\Gamma \setminus \{\mathbb{1}\}$. Then $a, b, c \in \{2, 4\}$. By [3, (3.2)] we may apply [4, Chapt. XIII, §3, Th. 2.3] and obtain the relation $a + b + c = 12$ which implies that $a = b = c = 4$. Finally, let $\Gamma \cong \mathbb{T}^2$. Then Γ contains an elementary abelian subgroup of rank two and thus $\mathscr{F}_\Gamma$ is one- or two-dimensional. If $\dim \mathscr{F}_\Gamma = 1$, using the notation from above, we obtain $a + b + c = 10$. Hence, we may assume that $a = 2$. Since Γ is a connected abelian group, it must fix the corresponding two-dimensional fix-double-loop pointwise, which is a contradiction. Hence we have $\dim \mathscr{F}_\Gamma = 2$.

1.3. Lemma. *If* $\dim \mathscr{F}_\Gamma = 0$, *then there exists an element* $c \in \mathscr{D} \setminus \mathscr{F}_\Gamma$ *with* $\dim c^\Gamma \leqslant 7$.

Proof. Suppose that $\dim c^\Gamma = 8$ for every element $c \in \mathscr{D} \setminus \mathscr{F}_\Gamma$. Then Γ acts transitively on the complement $\mathscr{D} \setminus \mathscr{F}_\Gamma$ by [3, (3.4)], the double loop $\mathscr{D}$ is a topological manifold homeomorphic to $\mathbb{R}^8$ ([20, 7.12] and [14, 5.2]), and the group Γ is a Lie group by [16, (6.3)]. Moreover, the orbit c^Γ is homeomorphic to the homogeneous space Γ/Γ_c by [3, (3.1)]. Since the complement $\mathscr{D} \setminus \mathscr{F}_\Gamma$ is simply connected by [1], this implies that Γ/Γ_c is simply connected as well. Applying Alexander duality to $\hat{\mathscr{D}}$ and $\hat{\mathscr{F}}_\Gamma$, we obtain for $0 \leqslant k \leqslant 7$ the isomorphisms

$$\tilde{\mathbf{H}}_k(\Gamma/\Gamma_c) \cong \tilde{\mathbf{H}}_k(c^\Gamma) \cong \tilde{\mathbf{H}}_k(\mathscr{D} \setminus \mathscr{F}_\Gamma) \cong \tilde{\mathbf{H}}_k(\hat{\mathscr{D}} \setminus \hat{\mathscr{F}}_\Gamma) \cong \tilde{\mathbf{H}}^{7-k}(\hat{\mathscr{F}}_\Gamma).$$

Note that for formulating Alexander duality for non-manifolds $\hat{\mathscr{F}}_\Gamma$ we have to take Alexander–Spanier cohomology groups, see e.g. [15, Th. 6.6, p. 222–223] or [7, VIII, 8.15]. Hence we have $\tilde{\mathbf{H}}_k(\Gamma/\Gamma_c) = 0$ if $0 \leqslant k \leqslant 6$ and $\mathbf{H}_7(\Gamma/\Gamma_c) = \bigoplus^\kappa \mathbb{Z}$, because $\hat{\mathscr{F}}_\Gamma$ is either finite or homeomorphic to the Cantor set, see [9, XI.1.5]. In any case, $\hat{\mathscr{F}}_\Gamma$ contains at least three path components (namely the singletons 0, 1, and ∞), which implies that $\kappa \geqslant 2$. Since the quotient Γ/Γ_c is simply connected, we may apply the Hurewicz isomorphy theorem (see [24, §7, Sect. 5, Th. 4]) and obtain the relation $\pi_7(\Gamma/\Gamma_c) \cong \mathbf{H}_7(\Gamma/\Gamma_c) = \bigoplus^{\kappa} \mathbb{Z}$. Thus, we have the following part of the long homotopy sequence

$$\pi_7(\Gamma) \xrightarrow{\alpha} \bigoplus^{\kappa} \mathbb{Z} \xrightarrow{\beta} \pi_6(\Gamma_c).$$

According to our general assumption that Γ is connected, the maximal compact subgroup K of Γ is a connected subgroup of $\mathrm{Spin}_3\,\mathbb{R}$, else Γ would contain an elementary abelian subgroup of rank two and hence $\mathscr{F}_\Gamma$ would be connected by Lemma 1.2. In particular, the group K is either trivial or isomorphic to one of the groups $\mathbb{T} \approx \mathbb{S}_1$ or $\mathrm{Spin}_3\,\mathbb{R} \approx \mathbb{S}_3$. By the Malcev–Iwasawa theorem [12, Th. 13, p. 549], the groups Γ and K are homotopy equivalent. This implies that the homotopy group $\pi_7(\Gamma)$ is finite, see [24, Chapt. 9, Sect. 7, Th. 7]. Moreover, the stabilizer Γ_c is also connected, because the quotient space Γ/Γ_c is simply connected [17, Chapt. 2, §8, Cor. 1]. As before, this implies that Γ_c is homotopy equivalent to $\mathbb{S}_1$, $\mathbb{S}_3$, or to a one-point space. Hence, the group $\pi_6(\Gamma_c)$ is also finite, see again [24, Chapt. 9, Sect. 7, Th. 7]. Now, the exactness of the sequence above implies that the kernel of β is infinite, which contradicts the fact that the image of α is finite.

2. Upper Bounds for $\dim \Gamma$

2.1. Lemma. *Let Φ and Ψ be non-trivial connected subgroups of Γ which centralize each other. Then, both groups are at most eight-dimensional or one of them is at most four-dimensional.*

Proof. If $\mathscr{F}_\Phi \neq \mathscr{F}_\Psi$, then $\dim \mathscr{F}_\Phi = \dim \mathscr{F}_\Psi = 4$ by Lemma 3.3, and hence by [10], both groups Φ and Ψ are at most eight-dimensional. Thus in the sequel we may assume that $\mathscr{F}_\Phi = \mathscr{F}_\Psi =: \mathscr{F}$.

Case 1. For each $c \in \mathscr{D} \setminus \mathscr{F}$ we have $\langle c^\Phi \rangle \neq \mathscr{D} \neq \langle c^\Psi \rangle$. Then $\dim \langle c^\Phi \rangle = \dim \langle c^\Psi \rangle = 4$ by [9, XI.8.5, XI.9.3] and, in particular, we have $\dim c^\Phi \leqslant 4$ and $\dim c^\Psi \leqslant 4$ for every $c \in \mathscr{D} \setminus \mathscr{F}$. Since the groups Φ and Ψ commute, the stabilizer Φ_c fixes the sub-double-loop $\langle c^\Psi \rangle$ pointwise and thus acts freely on the complement $\mathscr{D} \setminus \langle c^\Psi \rangle$. Hence we have $\dim \Phi_c = \dim d^{\Phi_c} + \dim \Phi_{c,d} \leqslant 4 + 0 = 4$ for some $d \in \mathscr{D} \setminus \langle c^\Psi \rangle$

and consequently $\dim \Phi \leqslant 8$ holds. Interchanging the roles of the groups Φ and Ψ this also gives $\dim \Psi \leqslant 8$.

Case 2. For all $c \in \mathscr{D} \backslash \mathscr{F}$ we have $\langle c^{\Phi} \rangle \neq \mathscr{D}$, and there is some $d \in \mathscr{D} \backslash \mathscr{F}$ with $\langle d^{\Psi} \rangle = \mathscr{D}$. Then the stabilizer Φ_d is trivial, since the groups Φ and Ψ commute. As in the first case we conclude that $\dim d^{\Phi} \leqslant 4$, and thus $\dim \Phi = \dim d^{\Phi} \leqslant 4$ holds.

Case 3. There exist elements $c, d \in \mathscr{D} \backslash \mathscr{F}$ such that $\langle c^{\Phi} \rangle = \mathscr{D} = \langle d^{\Psi} \rangle$. As in the second case we have $\Phi_d = \mathbb{1} = \Psi_c$, which immediately implies by [10] that $\dim \Phi = \dim d^{\Phi} \leqslant 8$ and $\dim \Psi = \dim c^{\Psi} \leqslant 8$.

We first study the case where Γ has a connected fix-double-loop.

2.2. Proposition. *If* $\dim \mathscr{F}_{\Gamma} \geqslant 1$, *then* $\dim \Gamma \leqslant 16$.

Proof. For $\dim \mathscr{F}_{\Gamma} \geqslant 2$ the assertion of the proposition is trivial by [10], since in this case the double loop $\mathscr{D}$ is generated by $\mathscr{F}_{\Gamma}$ and two additional elements of the complement $\mathscr{D} \backslash \mathscr{F}_{\Gamma}$. Thus let $\dim \mathscr{F}_{\Gamma} = 1$. Since by [9, XI.8.5, XI.9.3] we have $\dim \langle c^{\Gamma} \rangle \geqslant 4$ for all $c \in \mathscr{D} \backslash \mathscr{F}_{\Gamma}$, we shall distinguish two cases. If $\dim \langle c^{\Gamma} \rangle = 4$ for some $c \in \mathscr{D} \backslash \mathscr{F}_{\Gamma}$, then the group Γ acts on $\mathscr{H} := \langle c^{\Gamma} \rangle$ and we obtain

$$\dim \Gamma = \dim \Gamma|_{\mathscr{H}} + \dim \Gamma_{[\mathscr{H}]} \leqslant 4 + 8 = 12$$

by Lemma (3.2) of the appendix and [3]. Turning to the second case we have $\langle c^{\Gamma} \rangle = \mathscr{D}$ for all $c \in \mathscr{D} \backslash \mathscr{F}_{\Gamma}$. If there is an element $c \in \mathscr{D} \backslash \mathscr{F}_{\Gamma}$ such that $\dim \langle c \rangle \geqslant 4$, then there exists another element $d \in \mathscr{D}$ with $\mathscr{D} = \langle c, d \rangle$. Thus the stabilizer $\Gamma_{c,d}$ is trivial and this implies that $\dim \Gamma = \dim c^{\Gamma} + \dim d^{\Gamma_c} \leqslant 8 + 8 = 16$. Hence, we may assume that $\dim \langle c \rangle = 2$ for each $c \in \mathscr{D} \backslash \mathscr{F}_{\Gamma}$ and we conclude that $\dim c^{\Gamma} \leqslant 6$ holds for all $c \in \mathscr{D} \backslash \mathscr{F}_{\Gamma}$ by Lemma 1.1. Now, since $\mathscr{F}_{\Gamma}$ is connected and since $\mathscr{D}$ is generated by c^{Γ}, the double loop $\mathscr{D}$ is generated by $\mathscr{F}_{\Gamma}$ and at most three additional elements of the orbit c^{Γ}. This leads to the inequality $\dim \Gamma \leqslant 3 \dim c^{\Gamma} \leqslant 18$. Furthermore, we may assume that $\dim c^{\Gamma} = 6$, since we have $\dim \Gamma \leqslant 3 \cdot 5 = 15$ if $\dim c^{\Gamma} \leqslant 5$. So, for the rest of the proof we may assume that $17 \leqslant \dim \Gamma \leqslant 18$.

Case 1. Γ is semi-simple. Since there is no quasi-simple group of dimension 17 or 18, the inequality $17 \leqslant \dim \Gamma \leqslant 18$ implies that Γ is not quasi-simple. Let Z denote the center of Γ. By Lemma (3.3), either $\mathscr{F}_{\mathsf{Z}} = \mathscr{F}_{\zeta}$ for every $\zeta \in \mathsf{Z} \backslash \{\mathbb{1}\}$ or there is some element $\zeta_0 \in \mathsf{Z} \backslash \{\mathbb{1}\}$ with $\dim \mathscr{F}_{\zeta_0} = 4$. Suppose that Z has at least three elements. If $\mathscr{F}_{\mathsf{Z}} = \mathscr{F}_{\zeta}$

for all $\zeta \in Z \setminus \{\mathbb{1}\}$, by [9, XI.9.1, XI.9.3] this implies that $\dim \langle c^Z \rangle \geqslant 4$ for all $c \in \mathscr{D} \setminus \mathscr{F}_Z$, and hence $\dim \Gamma = \dim c^\Gamma + \dim \Gamma_c \leqslant 6 + 6 = 12$ holds. If, on the other hand, we have $\dim \mathscr{F}_\zeta = 4$ for some $\zeta \in Z \setminus \{\mathbb{1}\}$, then the group Γ leaves $\mathscr{F}_\zeta$ invariant, which implies that $\dim \Gamma = \dim \Gamma|\mathscr{F}_\zeta + \dim \Gamma[\mathscr{F}_\zeta] \leqslant 4 + 6 = 10$ by [3]. So let $|Z| \leqslant 2$. Then the group Γ is a Lie group which has a maximal torus subgroup of dimension at most two (toroidal rank at most two), see [9, XI.9.6]. Hence the group Γ has exactly two quasi-simple factors of toroidal rank one, because the universal covering of $SL_2\mathbb{R}$ is excluded by $|Z| \leqslant 2$. But semi-simple Lie groups of toroidal rank one are at most eight-dimensional, and so $\dim \Gamma \leqslant 16$ follows.

Case 2. Γ is not semi-simple. Then the group Γ contains a minimal connected closed abelian normal subgroup $\Xi \neq \mathbb{1}$ which is either compact or isomorphic as a topological group to a vector group $\mathbb{R}^t$. Since Ξ is connected, every orbit c^Ξ for $c \in \mathscr{D} \setminus \mathscr{F}_\Xi$ generates a sub-double-loop of dimension at least four, see [9, XI.8.5, XI.9.3]. If $\dim \langle c^\Xi \rangle = 4$, the stabilizer Γ_c acts on $\langle c^\Xi \rangle$, and we obtain $\dim \Gamma_c = \dim \Gamma_c|_{\langle c^\Xi \rangle} + \dim \Gamma_{[\langle c^\Xi \rangle]} \leqslant 4 + 6 = 10$ by [3] and Lemma (3.2). This implies that $\dim \Gamma \leqslant 10 + 6 = 16$.

So for the rest of the proof we may assume that $\langle c^\Xi \rangle = \mathscr{D}$. If Ξ is compact and hence central in Γ (see [11, (26.20)]), the stabilizer Γ_c is trivial and thus $\dim \Gamma \leqslant 6$ follows. Hence we may assume that Ξ is a vector group and that the stabilizer Γ_c acts effectively on Ξ. Let Γ_c^1 denote the connected component of Γ_c containing the identity and let $\Pi \leqslant \Xi$ denote a minimal Γ_c^1-invariant subspace of Ξ. Applying Lemma 3.3 to Ξ and Π, we obtain that either $\mathscr{F}_\Pi$ is four-dimensional and Γ_c^1 acts on $\mathscr{F}_\Pi$, or $c^\Pi \neq c$. If $\dim \mathscr{F}_\Pi = 4$, we conclude as before that $\dim \Gamma_c \leqslant 4 + 6 = 10$ and so $\dim \Gamma \leqslant 10 + 6 = 16$ holds. If $c^\Pi \neq c$, then we either have $\dim \langle c^\Pi \rangle = 4$, which implies that $\dim \Gamma = \dim c^\Gamma + \dim \Gamma_c \leqslant 6 + 10 = 16$, or $\langle c^\Pi \rangle = \mathscr{D}$. So it remains to study the case $\langle c^\Pi \rangle = \mathscr{D}$. Then Γ_c^1 acts effectively and irreducibly on the vector space Π, since Π is a minimal Γ_c^1-invariant subspace of Ξ. By [8, 19.14, 19.17], this implies that Γ_c^1 is a linear Lie group whose radical Δ is a closed connected subgroup of $\mathbb{C}^\times$ lying in the center of Γ_c^1. Let Ψ denote a Levi complement of Γ_c^1. By [9, XI.9.6] the dimension of the maximal torus of Γ_c^1 is at most two. Consequently, if $\Delta \cong \mathbb{C}^\times$ then the maximal torus of Ψ is at most one. Thus, by the classification of quasi-simple Lie groups we conclude that $\dim \Psi \leqslant 8$. Then $\dim \Gamma_c =$

$= \dim \Gamma_c^1 = \dim \Psi + \dim \Delta \leqslant 8 + 2 = 10$ which yields $\dim \Gamma \leqslant 16$. Hence we may assume that $\dim \Delta \leqslant 1$.

Now, the set $\Pi^* := \{\pi \in \Pi \mid c \in \mathscr{F}_\pi\}$ forms a subgroup of Π which is Γ_c^1-invariant, since for $\pi \in \Pi$, $c \in \mathscr{F}_\pi$, and $\gamma \in \Gamma$ we have $c^\gamma \in \mathscr{F}_{\pi^\gamma}$ and thus $c^\gamma = c$ implies that $\pi^\gamma \in \Pi^*$. Hence the set Π^* is Γ_c^1-invariant. Because of $c^\Pi \neq c$, we moreover have $\Pi^* \neq \Pi$. But the group Π is a minimal Γ_c^1-invariant subgroup. We conclude that $\Pi^* = \mathbb{1}$ and hence $c^\pi \neq c$ for all $\pi \in \Pi \setminus \{\mathbb{1}\}$. In particular, we have $\dim \langle c^{\mathsf{P}} \rangle \geqslant 4$ for every one-parameter subgroup P of Π and $s := \dim \Pi \leqslant 6$. Now, the stabilizer Γ_c^1 either acts transitively on Π, or there is a one-parameter subgroup P of Π such that $\dim \Gamma_c^1/\Theta < s$, where Θ is the centralizer of P in Γ_c^1. In the first case, Γ_c^1 acts transitively on the projective space $\mathscr{P}_{s-1}\mathbb{R}$ of all one-parameter subgroups of Π. If there is a one-parameter subgroup P with $\langle c^{\mathsf{P}} \rangle = \mathscr{D}$, then $\Theta = \mathbb{1}$, because Θ fixes c. This leads to

$$\dim \Gamma_c = \dim \Gamma_c^1/\Theta + \dim \Theta \leqslant s \leqslant 6$$

and hence we have $\dim \Gamma = \dim \Gamma_c + \dim c^\Gamma \leqslant 6 + 6 = 12$. So we may assume that $\langle c^{\mathsf{P}} \rangle < \mathscr{D}$ holds for all one-parameter subgroups P in Π. For the rest of the proof let us fix a one-parameter subgroup P in Π and let Θ be its centralizer in Γ_c^1. Since Θ acts freely on the complement $\mathscr{D} \setminus \langle c^{\mathsf{P}} \rangle$, we have $\dim \Theta = \dim d^\Theta + \dim \Theta_d \leqslant \dim c^\Gamma + 0 \leqslant 6$ for every $d \in c^\Gamma \setminus \langle c^{\mathsf{P}} \rangle$. Such an element d exists, because we have assumed that $\dim c^\Gamma = 6$. Altogether, we have

$$\dim \Gamma_c = \dim \Gamma_c^1/\Theta + \dim \Theta \leqslant s + 6. \qquad (*)$$

Thus, in the sequel we may suppose that $s \geqslant 5$. Since $\dim \Delta \leqslant 1$ and $11 \leqslant \dim \Gamma_c \leqslant 12$, the Levi-complement Ψ has one of the dimensions 10, 11, or 12. We will study each of these cases separately by using the classification of quasi-simple Lie groups and their representations.

a) $\dim \Psi = 10$. Then Ψ is quasi-simple and locally isomorphic to an orthogonal group $SO_{5,r}\mathbb{R}$. Since groups locally isomorphic to $SO_{5,r}\mathbb{R}$ have no irreducible representation of dimension 6, we conclude that $s \leqslant 5$. The stabilizer Γ_c has dimension at most 11, since $\dim \Delta \leqslant 1$. On the other hand, by what we have proved above we have $\dim \Gamma_c \geqslant 11$, and hence we know that $\dim \Gamma_c = 11$. Choose an element $d \in c^\Gamma \cap \langle c^{\mathsf{P}} \rangle \setminus \langle c \rangle$. If $d^{\Gamma_c} \subseteq \langle c^{\mathsf{P}} \rangle$, then $\dim d^{\Gamma_c} \leqslant \dim \langle c^{\mathsf{P}} \rangle = 4$ and we conclude that

$$\dim \Gamma_c = \dim d^{\Gamma_c} + \dim \Gamma_{c,d} \leqslant 4 + \dim \Gamma_{c,d}.$$

For every element $e \in c^{\Gamma} \setminus \langle c^{\mathsf{P}} \rangle$ we have $\mathscr{D} = \langle c, d, e \rangle$ and thus we conclude that $\dim \Gamma_{c,d} \leqslant \dim e^{\Gamma_{c,d}} \leqslant \dim e^{\Gamma} = \dim c^{\Gamma} = 6$. Consequently, we obtain $\dim \Gamma_c \leqslant 10$, which is a contradiction. Hence, we may assume that the orbit d^{Γ_c} is not contained in $\langle c^{\mathsf{P}} \rangle$ and therefore we may select an element $e \in d^{\Gamma_c} \setminus \langle c^{\mathsf{P}} \rangle$. As before we have $\mathscr{D} = \langle c, d, e \rangle$. Thus, the stabilizer $\Gamma_{c,d,e}$ is trivial, which implies that

$$11 = \dim \Gamma_c = \dim d^{\Gamma_c} + \dim \Gamma_{c,d} = \dim d^{\Gamma_c} + \dim e^{\Gamma_{c,d}}.$$

Since we have chosen the element d in the orbit c^{Γ}, we infer that $\dim d^{\Gamma_c} \leqslant \dim c^{\Gamma} = 6$ and just so from $e \in d^{\Gamma_c}$ we get $\dim e^{\Gamma_{c,d}} \leqslant \dim d^{\Gamma_c}$. By the equation above, this implies that $\dim e^{\Gamma_{c,d}} = 5$ (and $\dim d^{\Gamma_c} = 6$), and we infer that

$$\dim \Theta = \dim e^{\Theta} \leqslant \dim e^{\Gamma_{c,d}} = 5,$$

because the centralizer Θ fixes $\langle c^{\mathsf{P}} \rangle = \langle c, d \rangle$ pointwise. By inequality $(*)$, this yields $\dim \Gamma_c \leqslant s + 5 \leqslant 10$, which again is a contradiction.

b) $\dim \Psi = 11$. In this case, the group Ψ cannot be quasi-simple and, moreover, it is the product of an eight-dimensional quasi-simple group Ψ_1 and a three-dimensional quasi-simple group Ψ_2. Since Ψ is a linear group, a maximal torus subgroup of Ψ has dimension at least two. By Lemma 1.2, every involution in Ψ has a four-dimensional double loop of fixed elements. Select an involution ω in Ψ which is centralized by the factor Ψ_1. Thus, Ψ_1 leaves $\mathscr{F}_{\omega}$ invariant. Because Ψ_1 is quasi-simple, it must either act trivially or with a zero-dimensional kernel on $\mathscr{F}_{\omega}$. By Lemma 3.2 and [3], this implies that $\dim \Psi_1 \leqslant \max\{4, 6\} = 6$, which is a contradiction.

c) $\dim \Psi = 12$. Then $\Delta = \mathbb{1}$ and $\Gamma_c^1 = \Psi$ is the product of two six-dimensional quasi-simple groups Ψ_1 and Ψ_2, for else Ψ would contain a three-dimensional torus subgroup (note that Ψ is linear), which is impossible by [9, XI.9.6]. Moreover, by inequality $(*)$ we have $\dim \Theta = 6 = s$ and as mentioned above, Γ_c^1 acts transitively on $\mathscr{P}_{s-1}\mathbb{R}$. Since Ψ is linear, we can apply the classification of transitive connected linear groups acting on Graßmann manifolds, see [25], e.g. By this classification, $\Gamma_c^1 = \Psi$ has to be a quasi-simple group, which again is a contradiction.

We now turn to the general case where no restrictions on $\mathscr{F}_{\Gamma}$ are presumed. We start with a result about semi-simple groups.

2.3. Proposition. *If* Γ *is semi-simple, then* $\dim \Gamma \leqslant 16$.

Proof. The quotient $\Gamma^* := \Gamma/\mathsf{Z}(\Gamma)$ is a semi-simple Lie group, $\mathsf{Z}(\Gamma)$ is zero-dimensional, and a maximal compact Lie group K^* of Γ^* is covered by a Lie group $\tilde{\mathsf{K}}$ which is contained in the universal covering $\tilde{\Gamma}$ of Γ^*. In general, the group $\tilde{\mathsf{K}}$ need not be compact. So let $\tilde{C}$ denote a maximal compact subgroup of $\tilde{\mathsf{K}}$.

Case 1. Γ is quasi-simple. Then $\tilde{C}$ is a compact semi-simple Lie group, which is projected onto a compact semi-simple Lie subgroup C of Γ with $\dim C = \dim \tilde{C}$. Furthermore, the inequality $\dim C = \dim \tilde{C} \geqslant \dim \tilde{\mathsf{K}} - 1 = \dim \mathsf{K}^* - 1$ holds. If $\dim \Gamma^* \geqslant 14$ then $\dim \mathsf{K}^* \geqslant 6$ by the classification of quasi-simple Lie groups, and thus the group Γ contains a compact Lie group of dimension at least five. Consequently, the group Γ contains commuting involutions and the assertion follows from Lemma 1.2 and Proposition 2.2.

Case 2. Γ is semi-simple, but not quasi-simple. We write Γ as a product $\Gamma = \Phi \cdot \Psi \cdot \Delta$, where Φ and Ψ are non-trivial quasi-simple groups and Δ is a (possibly trivial) semi-simple group. By (2.1) we have $\dim \Gamma < \infty$, and thus we may assume that Φ is a quasi-simple factor of Γ of maximal dimension. The group Δ can be written as the product of at most two non-trivial semi-simple factors, because by Lemma 2.1 we have $\dim \Delta \leqslant 8$ (note that $\dim \Phi \cdot \Psi \geqslant 6 > 4$). Furthermore, we may assume that $\dim \Phi \geqslant 6$, since for $\dim \Phi \leqslant 3$ we would have $\dim \Gamma \leqslant 4 \cdot 3 = 12$. Now $\langle d^\Phi \rangle = \mathscr{D}$ must hold for all $d \in \mathscr{D} \backslash \mathscr{F}_\Phi$, because the quasi-simple group Φ acts on $\langle d^\Phi \rangle$ with a zero-dimensional kernel, i.e. the factor Φ induces a six-dimensional group on $\langle d^\Phi \rangle$, which is impossible by [3] and Lemma 3.2 if $\dim \langle d^\Phi \rangle = 4$. So we have $\langle d^\Phi \rangle = \mathscr{D}$. Set $\Lambda := \Psi \cdot \Delta$. Since Φ commutes with Λ and because of $\langle d^\Phi \rangle = \mathscr{D}$, the stabilizer Λ_d must be trivial. Consequently, we have $\dim \Lambda \leqslant 8$. Now assume that $\dim \Gamma \geqslant 17$. This implies that $\dim \Phi \geqslant 9$. Applying Lemma 2.1 to Φ and Λ, we conclude that $\dim \Lambda = 3$ and hence $\dim \Phi \geqslant 14$. Finally, choose an element $d \in \mathscr{D} \backslash \mathscr{F}_\Lambda$. Then $\dim \langle d^\Lambda \rangle \geqslant 4$. If $\langle d^\Lambda \rangle = \mathscr{D}$, then $\dim \Phi \leqslant 8$ (and hence $\dim \Gamma \leqslant 11$), since Λ and Φ commute. Finally, if $\dim \langle d^\Lambda \rangle = 4$, we obtain a contradiction as before.

2.4. Lemma. *A commutative group Γ is at most eight-dimensional. If Γ is an eight-dimensional commutative group, then it is isomorphic to* $\mathbb{R}^6 \times \mathbb{T}^2$.

Proof. Let $c \in \mathcal{D} \backslash \mathcal{F}_\Gamma$. Then $\dim \langle c^\Gamma \rangle \geqslant 4$. If $\langle c^\Gamma \rangle = \mathcal{D}$, then the stabilizer Γ_c is trivial, because Γ is commutative. Thus we have $\dim \Gamma \leqslant 8$. Suppose that $\dim \Gamma = 8$. Then the group Γ acts sharply transitively on the complement $\mathcal{D} \backslash \mathcal{F}_\Gamma$. Hence the group Γ is a Lie group, and the double loop $\mathcal{D}$ is a topological manifold (compare the proof of Lemma 1.3). Being a connected commutative Lie group, the group Γ is isomorphic to a product $\mathbb{R}^l \times \mathbb{T}^{8-l}$. Thus the homology groups $\mathbf{H}_n(\Gamma)$ of Γ vanish for $n \geqslant 2$. Applying Alexander duality to $\hat{\mathcal{D}}$ and $\hat{\mathcal{F}}_\Gamma$ and noting that $\Gamma/\Gamma_c \approx c^\Gamma = \mathcal{D} \backslash \mathcal{F}_\Gamma = \hat{\mathcal{D}} \backslash \hat{\mathcal{F}}_\Gamma$ holds by [3, (3.1)], this implies that the cohomology groups $\tilde{\mathbf{H}}^n(\hat{\mathcal{F}}_\Gamma)$ vanish for $0 \leqslant n \leqslant 5$. In particular, the dimension of $\mathcal{F}_\Gamma$ is at least six by Lemma 3.1. So we have $\mathcal{F}_\Gamma = \mathcal{D}$, because the dimension of a finite-dimensional locally compact connected double loop is either 1, 2, 4, or 8, see [9, XI.8.5] and compare also [14]. But then we have $\Gamma = \mathbb{1}$, a contradiction. Thus we have $\dim \Gamma \leqslant 7$ if $\langle c^\Gamma \rangle = \mathcal{D}$. Now let $\dim \langle c^\Gamma \rangle = 4$ for every $c \in \mathcal{D} \backslash \mathcal{F}_\Gamma$. Fix some element $d \in \mathcal{D} \backslash \mathcal{F}_\Gamma$ and set $\mathcal{H} := \langle d^\Gamma \rangle$. Choose an element $e \in \mathcal{D} \backslash \mathcal{H}$. Since $\dim \mathcal{H} = 4$, the group $\Gamma_d = \Gamma_{[\mathcal{H}]}$ operates freely on $\mathcal{D} \backslash \mathcal{H}$. In particular, the stabilizer Γ_d acts effectively on the four-dimensional sub-double-loop $\langle e^\Gamma \rangle$. This implies that $\dim \Gamma = = \dim d^\Gamma + \dim \Gamma_d \leqslant 4 + 4 = 8$, since the stabilizer Γ_d is at most four-dimensional by [3] and Lemma 3.2. Moreover, if $\dim \Gamma = 8$, Lemma 2.4 of [3] yields $\Gamma_d \cong \mathbb{R}^3 \times \mathbb{T}$. Interchanging the roles of the elements d and e, the same is true for the stabilizer Γ_e. Finally we have $\Gamma = \Gamma_d \times \Gamma_e$, since $\Gamma_d \cap \Gamma_e = \mathbb{1}$ and $\dim \Gamma_d = \dim \Gamma_e = 4$. This finishes our proof.

2.5. Proposition. *If* $\dim \mathsf{Z}(\Gamma) \geqslant 1$, *then* $\dim \Gamma \leqslant 16$.

Proof. Let $\mathsf{Z} := \mathsf{Z}(\Gamma)$ and choose an element $c \in \mathcal{D} \backslash \mathcal{F}_\mathsf{Z}$. Since $\dim \mathsf{Z} \geqslant 1$, the sub-double-loop $\langle c^\mathsf{Z} \rangle$ is of dimension at least four (see [9, XI.9.1., XI.9.3]). If $\langle c^\mathsf{Z} \rangle = \mathcal{D}$, the stabilizer Γ_c is trivial, which implies that $\dim \Gamma = \dim c^\Gamma \leqslant \dim \mathcal{D} = 8$. So we may assume that $\mathcal{H} := \langle c^\mathsf{Z} \rangle$ is four-dimensional. Then the stabilizer Γ_c acts freely on the complement $\mathcal{D} \backslash \mathcal{H}$, since $\Gamma_c = \Gamma_{[\mathcal{H}]}$. Thus we have $\dim \Gamma_c \leqslant 8$ and the assertion $\dim \Gamma = \dim c^\Gamma + \dim \Gamma_c \leqslant 8 + 8 = 16$ follows.

2.6. Theorem. *The automorphism group* Γ *of a locally compact connected eight-dimensional double loop* $\mathcal{D}$ *is at most* 16*-dimensional.*

Proof. By Proposition (2.3) we may assume that the group Γ contains a non-trivial connected commutative normal subgroup Ξ. If

Ξ is compact, it is contained in the center of Γ and the assertion of the theorem follows by Proposition 2.5. So, for the remainder of the proof we may assume that Ξ is not compact and hence is isomorphic to $\mathbb{R}^t$ for some $t > 0$. Moreover, by Proposition 2.2, we may assume that $\dim \mathscr{F}_\Gamma = 0$. Hence, by Lemma 1.3 we may select an element $c \in \mathscr{D} \backslash \mathscr{F}_\Gamma$ with $\dim c^\Gamma \leqslant 7$. Choose a minimal Γ_c^1-invariant subspace $\Pi \leqslant \Xi$. Using the arguments of [21, (3.3)], we may assume that the stabilizer Γ_c is a Lie group, since otherwise the dimension of $\langle c^\Gamma \rangle$ would be four and then $\dim \Gamma \leqslant 12$ by [3], because the group Γ leaves $\langle c^\Gamma \rangle$ invariant. Next, we may assume that Ξ moves the element c, since in the other case the fix-double-loop $\mathscr{F}_\Xi$ is four-dimensional by Lemma 3.3, and as before we conclude that $\dim \Gamma \leqslant 12$, because Γ leaves $\mathscr{F}_\Xi$ invariant.

In the following step we shall show that it suffices to consider the case where $\langle c^\Pi \rangle \neq \mathscr{D}$. Assume that $\langle c^\Pi \rangle = \mathscr{D}$ holds. Then Γ_c^1 acts effectively and irreducibly on the vector group Π. Consequently, the group Γ_c^1 is a linear Lie group with a radical of dimension at most two, see [8, 19.14, 19.17]. Now the assumption $\dim \Gamma \geqslant 17$ implies that $\dim \Gamma_c^1 \geqslant 10$, and therefore a Levi-complement Σ of Γ_c^1 is at least eight-dimensional. Being a linear semi-simple Lie group, Σ thus contains commuting involutions. Hence, we have $\dim \Gamma \leqslant 16$ by Lemma 1.2 and Proposition 2.2. So we may assume that $\langle c^\Pi \rangle \neq \mathscr{D}$. For the rest of the proof we also suppose that $\dim \Gamma \geqslant 17$. We shall distinguish two cases.

Case 1. The element c is fixed by the vector group Π. Then the fix-double-loop $\mathscr{F}_\Pi$ is four-dimensional by Lemma 3.3, since the element c is moved by the group Ξ. Set $\Pi^* := (\Xi \cap \Gamma_c^1)^1$. Since Γ_c^1 is closed in Γ and the group Π is contained in Π^* by assumption, we have $\Pi^* \cong \mathbb{R}^s$ for some $s > 0$. Moreover, the fix-double-loop $\mathscr{F}_{\Pi^*}$ is four-dimensional by Lemma 3.3, since $\mathscr{F}_{\Pi^*} \neq \mathscr{F}_\Xi$. The group Π^* is a normal subgroup of Γ_c^1, because Ξ is normal in Γ. Thus the product $\Xi\Gamma_c^1$ leaves the Baer double loop $\mathscr{F}_{\Pi^*}$ invariant and by Lemma 3.2 and [3] we obtain the inequality

$$\dim \Xi\Gamma_c^1 = \dim \Xi\Gamma_c^1 | \mathscr{F}_{\Pi^*} + \dim d^{\Xi\Gamma_c^1} \leqslant 4 + 7 = 11$$

for some element $d \in c^\Gamma \backslash \mathscr{F}_{\Pi^*}$. Using the (topological) isomorphism $\Xi\Gamma_c^1/\Xi \cong \Gamma_c^1/\Xi \cap \Gamma_c^1$ this yields the inequality

$$\dim \Gamma_c^1 = \dim \Xi\Gamma_c^1 - \dim \Xi + \dim(\Xi \cap \Gamma_c^1) \leqslant 11 - t + s.$$

Since $\dim \Gamma \geqslant 17$ and $\dim c^{\Gamma} \leqslant 7$, we have $\dim \Gamma_c^1 \geqslant 10$, and therefore $t - s \leqslant 1$ holds. If the orbit d^{Ξ} generates $\mathscr{D}$ for some element $d \in c^{\Gamma}$, then the stabilizer Ξ_d is trivial and thus $\Xi_c = \Xi_{d^\delta} = (\Xi_d)^\delta = \mathbb{1}$ for some automorphism $\delta \in \Gamma$ satisfying $d^\delta = c$. But this would contradict the fact that $\mathbb{1} \neq \Pi \leqslant \Xi_c$. Consequently, we have $\dim \langle d^{\Pi^*} \rangle = 4$ for every $d \in c^{\Gamma} \setminus \mathscr{F}_{\Pi^*}$. Furthermore, the group Π^* acts effectively on $\langle d^{\Pi^*} \rangle$, because $\mathscr{D}$ is generated by $\mathscr{F}_{\Pi^*}$ and the element d. Since Π^* is isomorphic to $\mathbb{R}^s$, this implies that $s \leqslant 3$ by [3, (2.4)]. Thus we have $t \leqslant 1 + s \leqslant 4$. Let P denote a closed one-parameter subgroup of Π^*. Since $\dim \mathscr{F}_{\Pi^*} = 4$, we have $\mathscr{F}_{\mathsf{P}} = \mathscr{F}_{\Pi^*}$ by Lemma (3.3). Therefore, the centralizer $\Theta := \mathrm{C}_{\Gamma}\mathsf{P}$ leaves $\mathscr{F}_{\mathsf{P}}$ invariant, and as before we infer that $\dim \Theta \leqslant 11$. Finally, considering the action of Γ on the space of all one-dimensional subspaces of Ξ we obtain that

$$\dim \Gamma \leqslant \dim \Theta + \dim \Xi \leqslant 11 + 4 \leqslant 15$$

which contradicts our assumption $\dim \Gamma \geqslant 17$.

Case 2. The element c is moved by the vector group Π. Then the double loop $\mathscr{H} := \langle c^{\Pi} \rangle$ is four-dimensional. Since Γ_c^1 acts on Π and fixes the element c, it leaves $\mathscr{H}$ invariant. This implies that

$$\dim \Gamma_c^1 = \dim(\Gamma_c^1)|_{\mathscr{H}} + \dim d^{\Gamma_c^1} \leqslant 4 + 7 = 11$$

for some $d \in c^{\Gamma} \setminus \mathscr{H}$. Let $\mathsf{N} \leqslant \Gamma_c^1$ denote the kernel of the action of Γ_c^1 on Π and set $\Lambda := \Gamma_{[\mathscr{H}]}$. Then we have $\mathsf{N} \leqslant \Lambda$. In the following we shall show that in fact $\mathsf{N} = \Lambda$. For this, we first verify that $c^\pi \neq c$ holds for all $\pi \in \Pi \setminus \{\mathbb{1}\}$. By Lemma 3.3, either $c^\pi \neq c$ or $\mathscr{F}_\pi = \mathscr{H}$ holds for any $\pi \in \Pi \setminus \{\mathbb{1}\}$. Consider the set $\Pi^* := \{\pi \in \Pi \mid \mathscr{H} \leqslant \mathscr{F}_\pi\}$. Using the same arguments as in the proof of Proposition 2.2 we infer that $\Pi^* = \mathbb{1}$ and hence $c^\pi \neq c$ holds for all $\pi \in \Pi \setminus \{\mathbb{1}\}$. Now we are able to prove the inclusion $\Lambda \subseteq \mathsf{N}$. For this, choose an element $\lambda \in \Lambda$. Suppose that there is an element $\pi \in \Pi$ with $\lambda^{-1}\pi\lambda \neq \pi$. Then $\mathbb{1} \neq \vartheta := \lambda^{-1}\pi\lambda\pi^{-1} \in \Pi$, and from the preceding arguments we infer that $c^\vartheta \neq c$. On the other hand, we have $c^\vartheta = c^{\lambda^{-1}\pi\lambda\pi^{-1}} = c^{\pi\lambda\pi^{-1}}$, and since $\lambda \in \Lambda$ and $c^\pi \in \mathscr{H}$, we conclude that $c \neq c^\vartheta = c^{\pi\lambda\pi^{-1}} = (c^\pi)^{\lambda\pi^{-1}} = (c^\pi)^{\pi^{-1}} = c$, which is a contradiction. This proves the equation $\mathsf{N} = \Lambda$.

Now, the quotient group $\Gamma_c^1/\mathsf{N} = \Gamma_c^1/\Lambda$ acts effectively and irreducibly on Π, since Π has been chosen to be minimal Γ_c^1-invariant. By [8, §19.14, §19.17] this implies that there is a semi-simple linear Lie group Σ (possibly $\Sigma = \mathbb{1}$) such that $\Sigma \leqslant \Gamma_c^1/\mathsf{N} \leqslant \Sigma\mathbb{C}^\times$. Thus either

$\Gamma_c^1/\mathsf{N} \leqslant \mathbb{R}$ or Γ_c^1/N contains a torus subgroup. In the second case, the involution ω which is contained in the torus subgroup induces an involution $\omega^* := \omega|_{\mathscr{H}}$ on $\mathscr{H}$ (note that $\mathsf{N} = \Lambda$). By [3, (3.3)], we have $1 \leqslant \dim \mathscr{F}_{\omega^*} \leqslant 2$ and the desired inequality $\dim \Gamma \leqslant 16$ follows from Proposition (2.2). So it remains to study the case where $\Gamma_c^1/\mathsf{N} \leqslant \mathbb{R}$. In this case, however, we immediately obtain that

$$\dim \Gamma_c \leqslant \dim \mathsf{N} + 1 = \dim \Lambda + 1 = \dim d^\Lambda + 1 \leqslant 7 + 1 = 8$$

for every element $d \in c^\Gamma \setminus \mathscr{H}$, and thus we have

$$\dim \Gamma = \dim c^\Gamma + \dim \Gamma_c \leqslant 7 + 8 = 15,$$

which finishes the proof.

3. Appendix

3.1. Lemma. *Let $\mathscr{D}$ be a locally compact connected double loop of arbitrary finite (covering) dimension n. Let U be a compact neighborhood in $\mathscr{D}$. Then the following statements hold:*

(a) $\dim U = \operatorname{ind} U = \operatorname{Ind} U = \dim_L U = n$ *for every principal ideal domain L, where* $\dim_L$ *denotes the cohomological dimension with coefficient domain L.*

(b) *The relation* $\dim(U \times X) = \dim U + \dim X$ *holds for every locally compact paracompact space X.*

Proof. The first assertion follows from Theorem 15.7 of [5] and from the fact that the covering dimension and the small and large inductive dimension coincide for a separable complete metric space, see [9, XI.1.2]. The second assertion follows from the equation $\dim U = \dim_{\mathbb{Z}_p} \mathscr{D}$ which holds for every prime number p by part (a), and from a theorem of Y. KODAMA in [13, p. 400].

3.2. Lemma. *Let $\mathscr{D}$ be a locally compact connected double loop of arbitrary finite dimension n. Let $\mathscr{H}$ be a closed sub-double-loop of $\mathscr{D}$ which is invariant under a locally compact automorphism group Γ of $\mathscr{D}$. The kernel of the action of Γ on $\mathscr{H}$ is denoted by Λ. Then the quotient group $\Delta = \Gamma/\Lambda$ of Γ is a topological transformation group on $\mathscr{H}$ with respect to the quotient topology τ_Q, and* $\dim(\Delta, \tau_Q) = \dim(\Delta, \tau_{co})$, *where τ_{co} denotes the compact-open topology on Δ with respect to the action of Δ on $\mathscr{H}$.*

Proof. Let $\Phi: \mathscr{H} \times \Gamma \to \mathscr{H}$ be the evaluation mapping. Since $\mathscr{H}$ is locally compact, the compact-open topology on Δ is the coarsest topology such that the mapping Φ is continuous. Thus the topology τ_Q on Δ is finer than the topology τ_{co}, since the group (Δ, τ_Q) evidently is a topological transformation group on $\mathscr{H}$. But his means that the identity mapping $\mathrm{id}:(\Delta, \tau_{co}) \to (\Delta, \tau_Q)$ is continuous. Since Γ is locally compact by [2], we may select a compact neighborhood U of the identity $\mathbb{1}$ in (Δ, τ_{co}). The restriction $\mathrm{id}|_U:(U, \tau_{co}) \to (U, \tau_Q)$ is a homeomorphism. Hence we conclude that

$$\begin{aligned}\dim(\Delta,\tau_{co}) &= \operatorname{ind}(\Delta, \tau_{co}) = \operatorname{ind}(U, \tau_{co}) = \operatorname{ind}(U, \tau_Q) = \\ &= \operatorname{ind}(\Delta, \tau_Q) = \dim(\Delta, \tau_Q),\end{aligned}$$

and the lemma is proved.

3.3. Lemma. *If* $\mathbb{1} \neq \Phi \trianglelefteq \Gamma$ *is a connected normal subgroup, then* $\mathscr{F}_\Phi = \mathscr{F}_\Gamma$ *or* $\mathscr{F}_\Phi$ *is four-dimensional. If* $1 \neq \gamma \in \Gamma$ *and* $\dim C \geqslant 1$, *where* $C := (C_\Gamma \gamma)^1$, *then* $\mathscr{F}_\gamma = \mathscr{F}_C$ *or* $\mathscr{F}_\gamma$ *is four-dimensional.*

Proof. Since Φ is a normal subgroup of Γ, the fix-double-loop $\mathscr{F}_\Phi$ is Γ-invariant. If $\mathscr{F}_\Phi \neq \mathscr{F}_\Gamma$, then Γ acts non-trivially on $\mathscr{F}_\Phi$. Because the group Γ is connected, this implies that $\dim \mathscr{F}_\Phi \geqslant 4$ and thus $\mathscr{F}_\Phi$ is four-dimensional. Similarly, the connected group C acts on $\mathscr{F}_\gamma$ and the claim follows as before.

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