Geometriae Dedicata **83:** 63–76, 2000. 63

Implicit Characterizations of Smooth Incidence Geometries

Dedicated to Helmut Salzmann on the occasion of his 70th birthday

RICHARD BÖDI[1] and STEFAN IMMERVOLL[2]
[1] *Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany. e-mail: richard.boedi@uni-tuebingen.de*
[2] *Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany. e-mail: stim@michelangelo.mathematik.uni-tuebingen.de*

(Received: 13 September 1999)

Communicated by K. Strambach

Abstract. We give a characterization of smooth stable and smooth projective planes in terms of submersion and transversality. Moreover, smooth affine translation planes are characterized by properties of their corresponding spreads, considered as subsets of the Grassmannian. The last section contains smoothness results about spherical Moebius planes. In particular, we establish smoothness properties of the classical Moebius plane.

Mathematics Subject Classification (2000): 51H25.

Key words: smooth incidence structures, smooth planes, Grassmann manifolds, transversality.

This paper is about incidence structures living on smooth manifolds. Although it was written with our minds set on the book *Compact Projective Planes* of H. Salzmann *et al.* [15], the paper should be readable without knowing this book (it is a pity if you don't, though). We will investigate how differentio-topological assumptions affect the underlying incidence geometries.

In the first part we will deal with rather general incidence structures living on manifolds. We will deduce implicit characterizations of smooth stable and smooth projective planes. The middle section assumes some background on translation planes and contains a characterization of smooth affine translation planes. The last part of this paper is devoted to non-linear incidence geometries, viz. to Moebius planes. In particular, we will derive some smoothness properties of the classical Moebius plane. Smoothness properties of generalized quadrangles will be investigated in a forthcoming paper by the second author.

In topological geometry, the most dominant object is that of a compact projective plane. From the incidence geometric point of view such a plane is a projective plane

$\mathcal{P} = (P, \mathcal{L}, \mathcal{F})$ with P as the set of points, $\mathcal{L}$ as the set of lines and $\mathcal{F} \subset P \times \mathcal{L}$ as the incidence relation (also known as the flag space). Additionally, the point and line spaces carry compact topologies such that the geometric operations of joining two distinct points with its unique line and of intersecting two distinct lines are continuous with respect to these topologies. Although this is quite a natural postulate, it sometimes appears to be rather unhandy. For example, it is not as easy as one might expect to verify that the classical projective planes $\mathcal{P}_2\mathbb{F}$ over one of the division algebras $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ satisfy the axioms of a topological plane, see section 14 of [15].

A characterization of compact projective planes omitting the continuity assumptions is due to Th. Grundhöfer, [5], 2.1. He showed that a projective plane with compact topologies on P and $\mathcal{L}$ is a topological plane if and only if the flag space $\mathcal{F}$ is a closed subset of $P \times \mathcal{L}$. Using this result it is immediate that the classical planes are indeed topological.

Having talked about topological geometry it is only a small step to *smooth geometry*. For a *smooth* projective plane we require the sets P and $\mathcal{L}$ to be smooth manifolds and the geometric operations to be smooth operations. Turning back to the flag space characterization of compact projective planes it is quite natural to ask if there is an analoguous condition for smooth projective planes. We will give an answer to this question in Corollary (1.4).

The word 'smooth' will be used as a synonym for k-times continuously differentiable for some k with $1 \leqslant k \leqslant \infty$. By a submanifold we always mean a smoothly embedded submanifold.

1. Implicit Characterizations of Stable Planes

Incidence structures are fundamental objects in geometry. Projective planes, circle planes, chain geometries, generalized polygons, to name a few, are prominent examples of incidence structures. Buekenhout's beautiful *Handbook of incidence geometry* [3] is a comprehensive source for such kinds of geometries.

For us, an *incidence structure* is a triple $(P, \mathcal{L}, \mathcal{F})$, where P and $\mathcal{L}$ are (disjoint) sets and $\mathcal{F} \subseteq P \times \mathcal{L}$. We interpret P as a set of 'points' and $\mathcal{L}$ as a set of 'lines'. The relation $\mathcal{F}$ is called the set of *flags*. We call a point p *incident* with a line L if and only if $(p, L) \in \mathcal{F}$.

Of course, incidence structures as described above are very general mathematical objects. In this section we want to show how mild differentio-topological assumptions can affect the geometry that is encoded in incidence structures. We will work with the following set of axioms:

DEFINITION. An incidence structure $\mathcal{I} = (P, \mathcal{L}, \mathcal{F})$ is called a *smooth generalized plane*, if there is an integer $l \in \mathbb{N}$ such that

(SGP1) P and $\mathcal{L}$ are $2l$-dimensional smooth manifolds, (SGP2) the flag space $\mathcal{F}$ is a $3l$-dimensional (smoothly embedded) submanifold of $P \times \mathcal{L}$, and the canonical projections $\pi_P : \mathcal{F} \to P : (p, L) \mapsto p$ and $\pi_{\mathcal{L}} : \mathcal{F} \to \mathcal{L} : (p, L) \mapsto L$ are submersions.

In Theorem (1.2) we will see that the notion of a 'smooth generalized plane' is adequate. The classical Moebius plane, in contrast, does not satfisfy these axioms, see Proposition (3.1). For $L \in \mathcal{L}$, the set $P_L := \pi_P(\pi_{\mathcal{L}}^{-1}(L))$ is called the *point row* associated to L. A point row is just the set of all points incident with a certain line. Dually, for some point $p \in P$ we call $\mathcal{L}_p := \pi_{\mathcal{L}}(\pi_P^{-1}(p))$ the *line pencil* through p. In incidence geometry it is often convenient to identify lines with their associated point rows. However, for our investigations, it is vital to distinguish between these two notions.

Note that axioms (SGP1) and (SGP2) are self-dual and hence smooth generalized planes satisfy the duality principle, i.e. every valid theorem remains true when the roles of the sets P and $\mathcal{L}$ are interchanged.

LEMMA 1.1. *Any point row (and any line pencil) of a smooth generalized plane $(P, \mathcal{L}, \mathcal{F})$ is a smoothly embedded l-dimensional submanifold of P (and $\mathcal{L}$), respectively.*

Proof. Because of our last remark, it suffices to prove the claim for line pencils only. So let $p \in P$. Then the inverse image $\pi_P^{-1}(p)$ is a smoothly embedded submanifold of $\mathcal{F}$, since by definition π_P is a submersion. Moreover, $\dim \pi_P^{-1}(p) = \dim \mathcal{F} - \dim P = 3l - 2l = l$. The map $\delta_p : \mathcal{L} \to P \times \mathcal{L} : L \mapsto (p, L)$ is smooth and we have $\delta_p \circ \pi_{\mathcal{L}}|_{\pi_P^{-1}(p)} = \mathrm{id}_{\pi_P^{-1}(p)}$. This proves that $\pi_{\mathcal{L}}|_{\pi_P^{-1}(p)} : \pi_P^{-1}(p) \to \mathcal{L}$ is a smooth embedding and thus $\mathcal{L}_p$ is a smoothly embedded l-dimensional submanifold of $\mathcal{L}$.

Remark. In order to prove that point rows are smoothly embedded l-dimensional submanifolds, the condition that $\pi_P : \mathcal{F} \to P$ is a submersion is superfluous.

DEFINITION. Two lines L_1 and L_2 of a smooth generalized plane are said to *intersect transversally in some point p*, if the associated point rows P_{L_1} and P_{L_2} intersect transversally in p as submanifolds of P, i.e. their tangent spaces in p span the tangent space T_pP, or, equivalently, the intersection of their tangent spaces in p is trivial. Transversality of line pencils is defined dually.

Transversality of point rows in one point has surprisingly strong consequences. In fact, the next theorem shows that under this condition the intersection map is 'locally' well defined and smooth. In more detail, we have

THEOREM 1.2. *Let $\mathcal{I} = (P, \mathcal{L}, \mathcal{F})$ be a smooth generalized plane. Assume that the lines $L_1, L_2 \in \mathcal{L}$ intersect transversally in $p \in P$. Then there are open neighborhoods U_i of L_i in $\mathcal{L}$, $i = 1, 2$, and V of p in P such that any two distinct lines $K_i \in U_i$, intersect* *in exactly one point $K_1 \wedge K_2 \in V$. Moreover, the intersection map $\wedge : U_1 \times U_2 \to V : (K_1, K_2) \mapsto K_1 \wedge K_2$ is smooth.*

Proof. Since the flag space $\mathcal{F}$ is a submanifold of $P \times \mathcal{L}$, for $i = 1, 2$ there is an open neighborhood V_i of (p, L_i) in $P \times \mathcal{L}$ as well as a smooth map $\psi_i : V_i \to \mathbb{R}^l$ of full rank everywhere, which vanishes exactly on the set $\mathcal{F} \cap V_i$. We set

$$\psi : V_1 \times V_2 \to \mathbb{R}^l \times \mathbb{R}^l : (x_1, x_2) \mapsto (\psi_1(x_1), \psi_2(x_2)),$$

$$\varphi : \mathcal{L} \times \mathcal{L} \times P \to (P \times \mathcal{L})^2 : (K_1, K_2, q) \mapsto (q, K_1, q, K_2)$$

and for $W := \varphi^{-1}(V_1 \times V_2)$ we put

$$F : W \to \mathbb{R}^l \times \mathbb{R}^l : (K_1, K_2, q) \mapsto \psi \circ \varphi(K_1, K_2, q).$$

By definition of F we have $F(K_1, K_2, q) = 0$ if and only if q is a common point of K_1 and K_2. In order to prove the assertions of the theorem using the implicit function theorem it suffices to check that the differential of the map

$$\{q \in P \mid (L_1, L_2, q) \in W\} \to \mathbb{R}^l \times \mathbb{R}^l : q \mapsto F(L_1, L_2, q)$$

is regular in p. So let $v \in \mathrm{T}_p P$ be in the kernel of this differential. The differentials $\mathrm{D}_{(p,L_1)}\psi_1$ and $\mathrm{D}_{(p,L_2)}\psi_2$ vanish exactly on $\mathrm{T}_{(p,L_1)}\mathcal{F}$ and $\mathrm{T}_{(p,L_2)}\mathcal{F}$, respectively. Using the chain rule and the definition of φ we thus get

$$(v, 0, v, 0) \in \mathrm{T}_{(p,L_1)}\mathcal{F} \times \mathrm{T}_{(p,L_2)}\mathcal{F}.$$

The projection $\pi_{\mathcal{L}}$ is a submersion, whence the subspace

$$\{(u, 0) \in \mathrm{T}_p P \times \mathrm{T}_{L_1}\mathcal{L} \mid (u, 0) \in \mathrm{T}_{(p,L_1)}\mathcal{F}\}$$

has dimension at most $3l - 2l = l$. Since $P_{L_1} \times \{L_1\} \subseteq \mathcal{F}$, we have $\mathrm{T}_p P_{L_1} \times \{0\} \subseteq \mathrm{T}_{(p,L_1)}\mathcal{F}$. Hence we conclude that

$$\{(u, 0) \in \mathrm{T}_p P \times \mathrm{T}_{L_1}\mathcal{L} \mid (u, 0) \in \mathrm{T}_{(p,L_1)}\mathcal{F}\} = \mathrm{T}_p P_{L_1} \times \{0\}.$$

From $(v, 0) \in \mathrm{T}_{(p,L_1)}\mathcal{F}$ we infer that $v \in \mathrm{T}_p P_{L_1}$ and analoguously we get $v \in \mathrm{T}_p P_{L_2}$. Because the lines L_1 and L_2 intersect transversally in p, this implies that $v = 0$, and we have proved the theorem.

A common notion in topological geometry is that of a stable plane, see R. Löwen, [9] and [10] for details. Before we are going to relate Theorem (1.2) to stable planes we need to give two more definitions.

DEFINITION. An incidence structure $(P, \mathcal{L}, \mathcal{F})$ is called a *linear space*, if any two distinct points $p, q \in P$ can be joined by exactly one line L, i.e. $(p, L), (q, L) \in \mathcal{F}$.

DEFINITION. A *smooth stable plane* $\mathcal{S}$ is a linear space $(P, \mathcal{L}, \mathcal{F})$ which satisfies the following axioms:

(SSP1) P and $\mathcal{L}$ are smooth manifolds such that the join map $\vee$ and the intersection map $\wedge$ are smooth, where the domain $\mathcal{O}$ of the intersection map is an open subset of $\mathcal{L} \times \mathcal{L}$ (*stability axiom*).

(SSP2) $\mathcal{S}$ contains four points such that any three of them do not lie on a common line.

A smooth stable plane $\mathcal{A}$ is called a *smooth affine plane* if it is an affine plane (from the incidence geometric point of view) and if the map which assigns to each line and each point p the parallel line through p is smooth. A *smooth projective plane* is a projective plane in the incidence geometric sense which is also a smooth stable plane.

COROLLARY 1.3. *Let $\mathcal{S} = (P, \mathcal{L}, \mathcal{F})$ be a smooth generalized plane as well as a linear space. If any two lines and any two line pencils intersect transversally, then $\mathcal{S}$ is a smooth stable plane.*

Proof. By our last theorem, the maximal domain of the intersection map is an open subset of $\mathcal{L} \times \mathcal{L}$ and both the join map and the intersection map are smooth. (Remember that the dual statement of that theorem is also true.) If $\mathcal{S}$ did not contain a quadrangle, the point space would consist of at most three point rows. This, in fact, is impossible due to Lemma (1.1) and axiom (SGP1).

If $\mathcal{S}$ is a stable plane, then the dimension assumptions in axioms (SGP1) and (SGP2) are automatically satisfied and the integer l is one of the numbers 1, 2, 4, or 8, see Löwen, [9] . By [1], p. 308, a smooth stable plane is a smooth generalized plane, and any two lines (any two line pencils) intersect transversally. Thus, Corollary (1.3) yields a characterization of smooth stable planes. For the particularly interesting case of smooth projective planes we formulate this characterization as another corollary.

COROLLARY 1.4. *Let $\mathcal{P} = (P, \mathcal{L}, \mathcal{F})$ be a projective plane. Then the following statements are equivalent:*

(i) *$\mathcal{P}$ is a smooth projective plane,*
(ii) *$\mathcal{P}$ is a smooth generalized plane such that any two distinct point rows and any two distinct line pencils intersect transversally.*

If P, $\mathcal{L}$, and $\mathcal{F}$ are compact, then the dimension assumptions in axioms (SGP1) and (SGP2) are automatically satisfied, see [5], 2.1, [15], 43.1. Our last result may considerably facilitate the verification that a given projective plane is smooth. For example, the proof that the classical projective plane $\mathcal{P}_2\mathbb{F}$ over one of the classical division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ is smooth is now immediate.

Corollary (1.4) is false without any assumptions on the flag space. There are examples of non-smooth projective planes whose point rows (and line pencils) are submanifolds, which intersect pairwise transversally in P (in $\mathcal{L}$), such that (SGPin2) is not satisfied. Moreover, these planes cannot be turned into smooth projective planes by changing the smooth structures on P and $\mathcal{L}$, see [8].

We will proceed with another characterization of smooth projective planes, which does not start with an abstract projective plane but which uses additional assumptions on the topology of the plane instead. The next theorem is the key to this characterization.

THEOREM 1.5. *Let $\mathcal{I} = (P, \mathcal{L}, \mathcal{F})$ be a smooth generalized plane such that any two lines and any two line pencils intersect transversally. Assume that P and $\mathcal{L}$ are compact and connected and that $\mathcal{F}$ is closed in $P \times \mathcal{L}$. Then there are positive integers m, n such that any two distinct points are joined by exactly m lines and any two distinct lines intersect in exactly n points. Furthermore, any two points rows (line pencils) are diffeomorphic.*

Proof. Since line pencils are l-dimensional manifolds with $l \geqslant 1$, there are two distinct intersecting lines L_1, L_2. By Theorem (1.2), the set of intersection points is discrete and also compact, because the point rows P_{L_1}, P_{L_2} are compact. Thus, we have a finite set $\{p_1, \ldots, p_n\}$ of intersection points. Choose pairwise disjoint open neighborhoods $V_1, \ldots, V_n$ of $p_1, \ldots, p_n$ in P. Again, by Theorem (1.2) there exist disjoint neighborhoods U_1 of L_1 and U_2 of L_2 in $\mathcal{L}$ such that any two lines $K_1 \in U_1$ and $K_2 \in U_2$ intersect in a unique point of V_k, $k = 1, \ldots, n$. By passing to smaller neighborhoods we may assume that any two lines $K_1 \in U_1$, $K_2 \in U_2$ intersect in exactly n points (use the compactness of P). Hence, each of the sets $\mathcal{O}_k := \{(L_1, L_2) \in \mathcal{L} * \mathcal{L} \mid |P_{L_1} \cap P_{L_2}| = k\}$, $k = 1, 2, \ldots$ is open in $\mathcal{L} * \mathcal{L}$. Here, $\mathcal{L} * \mathcal{L} := (\mathcal{L} \times \mathcal{L}) \bmod \mathbb{Z}_2$ denotes the *symmetric product* of $\mathcal{L}$, where $\mathbb{Z}_2$ acts on $\mathcal{L} \times \mathcal{L}$ by interchanging the coordinates. Obviously, the set $\mathcal{O}_0 = \{(L_1, L_2) \in \mathcal{L} * \mathcal{L} \mid P_{L_1} \cap P_{L_2} = \emptyset\}$ is open, too. The connected set $\{(K_1, K_2) \in \mathcal{L} * \mathcal{L} \mid K_1 \neq K_2\}$ is covered by the family of pairwise disjoint sets $\mathcal{O}_k$. We conclude that only one of the sets $\mathcal{O}_k$ $(k = 0, 1, 2, \ldots)$ is non-empty, namely $\mathcal{O}_n$. This proves that any two lines intersect in exactly n points. By duality, there is a positive integer m such that two distinct line pencils intersect in exactly m lines. Equivalently, any two distinct points are joined by exactly m lines.

In order to prove the last assertion, we use the fact that the projection $\pi_P : \mathcal{F} \to P$ is a smooth locally trivial fibering (by a Theorem of Ehresmann, see [2], (8.12)). Since $\pi_P^{-1}(p) = \mathcal{L}_p$, we infer that any two line pencils are diffeomorphic. Analogously, any two point rows are diffeomorphic.

Remark. It would be interesting to know whether $n, m \neq 1$ can actually occur.

COROLLARY 1.6. *Assume that $\mathcal{I} = (P, \mathcal{L}, \mathcal{F})$ satisfies the conditions of the Theorem (1.5). If there are two lines whose intersection consists of at most one point,* *or if there are two points which are joined by at most one line, then $\mathcal{I}$ is a smooth projective plane.*

Remark. This corollary can be used to construct examples of smooth projective planes.

2. Smooth Affine Translation Planes

In this section we proceed with the study of smooth incidence geometries in the special case of affine translation planes. Smooth translation planes have been studied by Th. Grundhöfer, H. Hähl [6], [7] and J. Otte, [13], [14], who proved that there are no non-classical projective translation planes.

Any affine translation plane $\mathcal{A} = (\mathbb{R}^{2l}, \mathcal{L}, \subseteq)$ with point set $\mathbb{R}^{2l}$ can be modelled by l-dimensional affine subspaces, i.e. $\mathcal{L}$ consists of a suitable family of l-dimensional affine subspaces of $\mathbb{R}^{2l}$, which is uniquely determined by its subfamily $\mathcal{S}$ (called a *spread*) of l-dimensional vector subspaces. Axiomatically, a spread of some vector space V of dimension $2l$ consists of l-dimensional vector subspaces of V that intersect pairwise trivially and whose union covers V. According to Löwen, [11], such an affine translation plane is topological if and only if the spread $\mathcal{S}$ is a closed subset of $\mathrm{G}(l, 2l)$. Here, $\mathrm{G}(l, 2l)$ denotes the Grassmannian of l-dimensional vector subspaces of $\mathbb{R}^{2l}$.

In this section we want to give a similar characterization for smooth affine translation planes. J. Otte ([13], 5.14) showed that an affine translation plane is smooth if and only if the map $\rho : \mathbb{R}^{2l} \setminus \{0\} \to \mathrm{G}(l, 2l)$, which assigns to any point $s \in \mathbb{R}^{2l} \setminus \{0\}$ the unique spread element $S \in \mathcal{S}$ containing s, is smooth. Furthermore, he verified that in this case $\mathcal{S}$ is in fact a submanifold of the Grassmannian $\mathrm{G}(l, 2l)$, see [13], 5.15.

For $s \in \mathbb{R}^{2l} \setminus \{0\}$ we denote by $\mathrm{G}_s(l, 2l)$ the subset of $\mathrm{G}(l, 2l)$ which consists of all l-dimensional vector subspaces that contain s. Using Grassmann coordinates, it is easy to see that $\mathrm{G}_s(l, 2l)$ is an $(l^2 - l)$-dimensional submanifold of $\mathrm{G}(l, 2l)$. Note that $\mathrm{G}_s(l, 2l)$ is homeomorphic to $\mathrm{G}_s(l-1, 2l-1)$.

The following characterization theorem will be proved at the end of this section.

THEOREM 2.1. *For an affine translation plane $\mathcal{A} = (\mathbb{R}^{2l}, \mathcal{L}, \subset)$ the following assertions are equivalent:*

(i) *$\mathcal{A}$ is a smooth affine translation plane.*
(ii) *The spread associated to $\mathcal{A}$ is a submanifold of the Grassmannian* $\mathrm{G}(l, 2l)$*, which intersects* $\mathrm{G}_s(l, 2l)$ *transversally for any $s \in \mathbb{R}^{2l} \setminus \{0\}$.*

In order to show that assertion (i) is a consequence of (ii) we will give a geometric proof in the spirit of the preceding section, which is almost independent of Otte's methods. However, Theorem (2.1) can easily be deduced from 6.12, 1.15, 1.11 and 1.8 of [13]. We proceed with some lemmas.

LEMMA 2.2. *The set* $\mathcal{F} := \{(s, S) \in (\mathbb{R}^{2l} \setminus \{0\}) \times \mathrm{G}(l, 2l) \mid s \in S\}$ *is a submanifold of* $\mathbb{R}^{2l} \times \mathrm{G}_s(l, 2l)$.

Proof. Let $\pi_S : \mathbb{R}^{2l} \to S$ be the orthogonal projection onto the subspace S of $\mathbb{R}^{2l}$. It suffices to check that the map

$$\pi : \mathbb{R}^{2l} \times \mathrm{G}(l, 2l) \to \mathbb{R}^{2l} \times \mathrm{G}(l, 2l) : (s, S) \longmapsto (\pi_S(s), S)$$

is smooth, because $\mathcal{F}$ is an open subset of the image of π and $\pi^2 = \pi$, see [2], p. 53. Choose $X, Y \in \mathrm{G}(l, 2l)$ such that X is orthogonal to Y in Euclidean space $\mathbb{R}^{2l}$. We identify $\mathbb{R}^{2l}$ with the orthogonal sum $X \perp Y$, and the Euclidean spaces X, Y with the Euclidean space $\mathbb{R}^l$. We introduce Grassmann coordinates with respect to X and Y. According to the identifications above, the Grassmann coordinates of a vector subspace $Z \in \mathrm{G}(l, 2l)$ with $Z \cap Y = \{0\}$ correspond to some matrix A_Z defined by the equation $Z = \{(x, A_Z x) \in \mathbb{R}^l \perp \mathbb{R}^l \mid x \in \mathbb{R}^l\}$. The orthogonal subspace $Z^\perp$ is given by $\{(-A_Z^t x, x) \in \mathbb{R}^l \perp \mathbb{R}^l \mid x \in \mathbb{R}^l\}$. The vector subspace X corresponds to the matrix $A_X = 0$. Let U be the open neighborhood of X in $\mathrm{G}(l, 2l)$ consisting of all elements $Z \in \mathrm{G}(l, 2l)$ such that the matrix $\mathrm{id} + A_Z^t A_Z$ is invertible. Select $s = (p, q) \in \mathbb{R}^l \perp \mathbb{R}^l$ and $Z \in U$. Then $\pi_Z(s)$ is the unique intersection point of $(p, q) + Z^\perp$ and Z and thus is determined by the equation $(x_1, A_Z x_1) = (-A_Z^t x_2, x_2) + (p, q)$, $x_1, x_2 \in \mathbb{R}^l$. A straightforward calculation shows that x_1 is given by $x_1 = (\mathrm{id} + A_Z^t A_Z)^{-1}(p + A_Z^t q)$. Hence, the image $\pi_Z(s) = (x_1, A_Z x_1)$ depends smoothly on $s \in \mathbb{R}^l \perp \mathbb{R}^l$ and on $Z \in U$. Since $X \in \mathrm{G}(l, 2l)$ was chosen arbitrarily, we have proved that the map $\mathbb{R}^{2l} \times \mathrm{G}(l, 2l) : (s, S) \longmapsto \pi_S(s)$ is smooth, whence π is smooth.

LEMMA 2.3. *The canonical projection* $\tau : \mathcal{F} \to \mathbb{R}^{2l} \setminus \{0\} : (s, S) \longmapsto s$ *is a submersion and thus the dimension of* $\mathcal{F}$ *is* $l^2 + l$.

Proof. The set

$$\mathcal{O} := \left\{(y_1, \dots, y_l) \in \left(\mathbb{R}^{2l}\right)^l \mid y_1, \dots, y_l \text{ linearly independent}\right\}$$

is an open subset of $\left(\mathbb{R}^{2l}\right)^l$. Consider the map

$$\sigma : (\mathbb{R}^l \setminus \{0\}) \times \mathcal{O} \to \mathcal{F} : (s_1, \dots, s_l, y_1, \dots, y_l) \longmapsto (s_1 y_1 + \dots + s_l y_l, \langle y_1, \dots, y_l \rangle),$$

where $\langle y_1, \dots, y_l \rangle$ denotes the linear span of $y_1, \dots, y_l$. In order to prove the first statement we only need to verify that the composition

$$\tau \circ \sigma : (\mathbb{R}^l \setminus \{0\}) \times \mathcal{O} \to \mathbb{R}^{2l} : (s_1, \dots, s_l, y_1, \dots, y_l) \longmapsto s_1 y_1 + \dots + s_l y_l$$

is a submersion. Let $x = (t_1, \dots, t_l, x_1, \dots, x_l) \in (\mathbb{R}^l \setminus \{0\}) \times \mathcal{O}$. Without loss of generality we may assume that $t_1 \neq 0$. The differential of the map

$$\mathbb{R}^{2l} \to \mathbb{R}^{2l} : y_1 \mapsto t_1 y_1 + t_2 x_2 + \ldots + t_l x_l$$

in $x_1 \in \mathbb{R}^{2l}$ has rank $2l$, whence the differential of $\tau \circ \sigma$ in x has also rank $2l$. Since we did not impose any restriction on x, this proves the first assertion. As $\tau^{-1}(s) = \mathrm{G}_s(l, 2l)$ is of dimension $l^2 - l$, we end up with

$$\dim \mathcal{F} = \dim \mathbb{R}^{2l} + \dim \mathrm{G}_s(l, 2l) = 2l + (l^2 - l) = l^2 + l.$$

LEMMA 2.4. *Let* $\mathcal{A} = (\mathbb{R}^{2l}, \mathcal{L}, \subset)$ *be an affine translation plane with associated spread* $\mathcal{S}$. If $\mathcal{S}$ is a submanifold of the Grassmannian $\mathrm{G}(l, 2l)$, then $\dim \mathcal{S} = l$.

Proof. We select two distinct spread elements $X, Y \in \mathcal{S}$ and introduce Grassmann coordinates with respect to X and Y. Let W be some line parallel to Y which intersects X in $p \in X$. Identifying $\mathbb{R}^{2l}$ with $X \oplus Y$, the intersection of some spread element $S \in \mathcal{S} \setminus \{Y\}$ with W is given by $(p, A_S p)$, where A_S is the matrix of Grassmann coordinates of S. Since $\mathcal{A}$ is an affine plane, the map $\gamma : \mathcal{S} \setminus \{Y\} \to W : S \mapsto (p, A_S p)$ is a smooth bijection. By Sard's Theorem there is a regular value $w \in W$. Thus, $\dim \mathcal{S} = \dim W + \dim \gamma^{-1}(w) = l + 0$. Alternatively, you can use invariance of domain, cp. [4], Ex. (18.10) or [12], Ex. 6.8, p. 217.

Remark. If we assume that $\mathcal{S}$ is closed in $\mathrm{G}(l, 2l)$, the assertion of the last lemma is a consequence of Löwen [11].

Now we are able to tackle the

Proof of Theorem (2.1). First, we assume that (ii) holds. By Otte [13], 5.11, 5.12 and 5.14, we have to show that the map $\rho : \mathbb{R}^{2l} \setminus \{0\} \to \mathrm{G}(l, 2l)$ defined above is smooth. We do this by using again the implicit function theorem. Let $s \in \mathbb{R}^{2l} \setminus \{0\}$ and $S \in \mathcal{S}$ with $s \in S$. Since $\mathcal{S}$ is a smooth l-dimensional submanifold of $\mathrm{G}(l, 2l)$ there is a smooth regular map $\varphi : U \to \mathbb{R}^{l^2 - l}$ defined on some open neighborhood U of S in $\mathrm{G}(l, 2l)$, such that $\varphi(X) = 0$ if and only if $X \in \mathcal{S} \cap U$. Because $\mathcal{F}$ is a $(l^2 + l)$-dimensional submanifold of $\mathbb{R}^{2l} \setminus \{0\} \times \mathrm{G}(l, 2l)$, the same argument yields a smooth regular map $\psi : V \to \mathbb{R}^l$ defined on some open neighborhood V of (s, S) in $\mathbb{R}^{2l} \setminus \{0\} \times \mathrm{G}(l, 2l)$, which vanishes exactly on $\mathcal{F} \cap V$. Moreover, we may assume that $X \in U$ for any $(p, X) \in V$. The smooth map

$$F : V \to \mathbb{R}^{l^2 - l} \times \mathbb{R}^l : (p, X) \mapsto (\varphi(X), \psi(p, X))$$

vanishes at (p, X) if and only if $X = \rho(p)$. Hence, by the implicit function theorem, it suffices to verify that the differential of the locally defined map $X \mapsto F(s, X)$ is regular at S. Let $v \in \mathrm{T}_S \mathrm{G}(l, 2l)$ be in the kernel of this differential. Recalling that $\mathrm{D}_S \varphi$ vanishes exactly on $\mathrm{T}_S \mathcal{S}$ and $\mathrm{D}_{(s,S)} \psi$ vansishes exactly on $\mathrm{T}_{(s,S)} \mathcal{F}$, we infer that

$v \in \mathrm{T}_S\mathcal{S}$ and $(0, v) \in \mathrm{T}_{(s,S)}\mathcal{F}$. Since the canonical projection $\tau : \mathcal{F} \to \mathbb{R}^{2l}$ is a submersion, the subspace $\{(0, u) \in \mathrm{T}_{(s,S)}\mathcal{F}\}$ of $\mathrm{T}_{(s,S)}\mathcal{F}$ has dimension at most $(l^2 + l) - 2l = l^2 - l$. On the other hand, $\{0\} \times \mathrm{T}_S\mathrm{G}_s(l, 2l)$ is contained in this subspace because of $\{s\} \times \mathrm{G}_s(l, 2l) \subseteq \mathcal{F}$. Thus, from $\dim \mathrm{G}_s(l, 2l) = l^2 - l$ we infer that

$$\{(0, u) \in \mathrm{T}_{(s,S)}\mathcal{F}\} = \{0\} \times \mathrm{T}_S\mathrm{G}_s(l, 2l)$$

and hence $v \in \mathrm{T}_S\mathrm{G}_s(l, 2l)$. By hypothesis, the manifolds $\mathcal{S}$ and $\mathrm{G}_s(l, 2l)$ intersect transversally in S, which implies that $v = 0$. This proves (i).

Conversely, assume that (i) holds. By Otte [13], 5.15, we know that $\mathcal{S}$ is a submanifold of $\mathrm{G}(l, 2l)$. Select $s \in \mathbb{R}^{2l} \setminus \{0\}$ and $S \in \mathcal{S}$ with $s \in S$. We have to show that $\mathrm{G}_s(l, 2l)$ and $\mathcal{S}$ intersect transversally in S. Let $Y \in \mathcal{S} \setminus \{S\}$. By introducing Grassmann coordinates with respect to S and Y, any element $Z \in \mathrm{G}_s(l, 2l)$ with $Z \cap Y = \{0\}$ corresponds to some linear map $A_Z : X \to Y$ that satisfies $A_Z(s) = 0$. The tangent space T of the submanifold $\{A_Z \mid Z \in \mathcal{S} \setminus \{Y\}\}$ of $\mathbb{R}^{l\times l}$ in the point A_S is contained in $\mathrm{GL}_l\mathbb{R} \cup \{0\}$, see [13], 1.7 and 5.14. This shows that the tangent space of $\{A_Z \mid Z \in \mathrm{G}_s(l, 2l), Z \cap Y = \{0\}, A_Z(s) = 0\}$ in A_S has trivial intersection with T. Hence we get $\mathrm{T}_S\mathcal{S} \cap \mathrm{T}_S\mathrm{G}_s(l, 2l) = \{0\}$. Because of $\dim \mathcal{S} = l$ and $\dim \mathrm{G}_s(l, 2l) = l^2 - l$ we have proved (ii).

3. Smoothness Properties of Moebius Planes

In our last section we want to leave 'linear' incidence geometries and enter the area of the so-called *circle geometries*. Here, we want to focus on *Moebius planes*. The classical example $\mathcal{M}_{\mathrm{cl}}$ of a Moebius plane is the 2-sphere $\mathbb{S}_2$ together with all plane sections of $\mathbb{S}_2$ that contain more than one point. In general, a Moebius plane is defined as follows:

DEFINITION. An incidence structure $\mathcal{M} = (S, \mathcal{K}, \mathcal{F})$, where $\mathcal{F} \subseteq S \times \mathcal{K}$, is called a *Moebius plane*, if and only if the following axioms are satisfied:

(MP1) For any three distinct points there exists a unique circle incident with these three points.
(MP2) For any two distinct points p, q and any circle K incident with p but not with q there exists a unique circle L through q whose intersection with K consists of p only.
(MP3) There exists some circle that contains three but not all points, and every circle is incident with some point.

Analogously as for topological linear incidence geometries like stable planes, we require for a topological Moebius plane that – among other conditions – the map

$$\alpha : \{(x, y, z) \in S^3 \mid |\{x, y, z\}| = 3\} \to \mathcal{K},$$

which assigns to any three pairwise distinct points their unique joining circle, and the map

$$\beta : \{(K, L) \in \mathcal{K}^2 \mid S_K \cap S_L \neq \emptyset\} \to S * S,$$

which maps each pair of distinct intersecting circles to their intersection points, are continuous, compare Wölk [17]. Here, $S * S$ is the symmetric product (see the proof of Theorem (1.5)) and S_X denotes the set of points incident with the circle X. The aim of this section is to establish properties of Moebius planes which imply that the above maps are smooth in the sense explained in Theorems (3.3) and (3.4).

According to K. Strambach [16], the space of circles of a topological Moebius plane is homeomorphic to $\mathcal{P}_3\mathbb{R}$ with a closed ball removed, if the point space is homeomorphic to $\mathbb{S}_2$. Note that the latter assumption is satisfied for any topological Moebius plane with a locally compact, connected point space. However, we will not prove this in order to proceed with our ideas.

The conditions in Theorems (3.3) and (3.4) are motivated by properties of the classical Moebius plane $\mathcal{M}_{\text{cl}}$. The point space S of $\mathcal{M}_{\text{cl}}$ is $\mathbb{S}_2$. To each circle in the unit sphere $\mathbb{S}_2 \subset \mathbb{R}^3$ there corresponds a unique plane whose distance from the origin is less than one. Such planes are described by equations of the form $a_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0$, where $a_0^2 < a_1^2 + a_2^2 + a_3^2$. Hence, we may identify the space $\mathcal{K}$ with the set

$$\{(a_0, a_1, a_2, a_3) \in \mathcal{P}_3\mathbb{R} \mid a_0^2 < a_1^2 + a_2^2 + a_3^2\}.$$

In this way, $\mathcal{K}$ is endowed with a smooth structure. In the following proposition we collect some differentio-topological properties of $\mathcal{M}_{\text{cl}}$.

PROPOSITION 3.1. *For the classical Moebius plane $\mathcal{M}_{\text{cl}} = (S, \mathcal{K}, \mathcal{F})$ the following statements hold:*

(CMP1) *The flag space $\mathcal{F}$ is a 4-dimensional submanifold of $S \times \mathcal{K}$.*
(CMP2) *The canonical projections $\pi_S : \mathcal{F} \to S : (p, K) \mapsto p$ and $\pi_{\mathcal{K}} : \mathcal{F} \to \mathcal{K} : (p, K) \mapsto K$ are submersions.*
(CMP3) *For any two circles $K, L \in \mathcal{K}$ with $|S_K \cap S_L| = 2$ the intersection of S_K and S_L is transversal.*
(CMP4) *For any point $p \in S$ the set $\mathcal{K}_p := \{K \in \mathcal{K} \mid (p, K) \in \mathcal{F}\}$ is a 2-dimensional submanifold of $\mathcal{K}$.*
(CMP5) *Let $p, q, r \in S$ be three pairwise distinct points and let K be the unique circle incident with these three points. Then $\mathrm{T}_K\mathcal{K}_p \cap \mathrm{T}_K\mathcal{K}_q \cap \mathrm{T}_K\mathcal{K}_r = \{0\}$.*

Proof. (CMP1) The smooth map

$$F : \mathbb{S}_2 \times \mathcal{K} \to \mathbb{R} : (x_1, x_2, x_3, a_0, a_1, a_2, a_3) \mapsto a_0 + a_1x_1 + a_2x_2 + a_3x_3$$

has full rank and $F(x_1, x_2, x_3, a_0, a_1, a_2, a_3)$ vanishes if and only if the point

$(x_1, x_2, x_3) \in \mathbb{S}_2$ lies on the circle that corresponds to $(a_0, a_1, a_2, a_3) \in \mathcal{K}$. Thus, the flag space $\mathcal{F}$ is a submanifold of $\mathbb{S}_2 \times \mathcal{K}$ of dimension $5 - 1 = 4$.
(CMP2) We consider the tangent space $\mathrm{T}_{(x,a)}\mathcal{F}$ for some point

$$(x, a) = (x_1, x_2, x_3, a_0, a_1, a_2, a_3) \in \mathcal{F} \subset \mathbb{S}_2 \times \mathcal{K},$$

where $\mathbb{S}_2 \subset \mathbb{R}^3$ and (a_0, a_1, a_2, a_3) are homogeneous coordinates. By symmetry we may assume that $a_3 = 1$, and in this way we introduce inhomogeneous coordinates in some neighborhood of (x, a). Then any element of $\mathcal{F}$ in this neighborhood corresponds to some vector $(y_1, y_2, y_3, b_0, b_1, b_2) \in \mathbb{S}_2 \times \mathbb{R}^3$ with $b_0 + b_1 y_1 + b_2 y_2 + y_3 = 0$. The tangent space $\mathrm{T}_{(x,a)}\mathcal{F}$ is given by

$$\begin{aligned} \mathrm{T}_{(x,a)}\mathcal{F} &= \ker \mathrm{D}_{(x,a)}F \\ &= \{(u_1, u_2, u_3, v_0, v_1, v_2) \in x^{\perp} \times \mathbb{R}^3 \mid a_1 u_1 + a_2 u_2 + u_3 + v_0 + x_1 v_1 + x_2 v_2 \\ &\qquad = 0\}, \end{aligned}$$

where $x^{\perp}$ is the subspace of $\mathbb{R}^3$ of all vectors perpendicular to x. For $(u_1, u_2, u_3) \in x^{\perp}$ there is $(v_0, v_1, v_2) \in \mathbb{R}^3$ such that $a_1 u_1 + a_2 u_2 + u_3 + v_0 + x_1 v_1 + x_2 v_2 = 0$. Hence, the projection π_S is a submersion.

We turn to the projection $\pi_{\mathcal{K}}$. Let $(v_0, v_1, v_2) \in \mathbb{R}^3$. We are looking for some element $(u_1, u_2, u_3) \in x^{\perp}$ such that $\langle (a_1, a_2, 1), (u_1, u_2, u_3) \rangle = -(v_0 + x_1 v_1 + x_2 v_2)$, which exists if the vectors x and $(a_1, a_2, 1)$ are linearly independent. Indeed, this is the case. Otherwise, the plane given by the equation $a_0 + a_1 y_1 + a_2 y_2 + y_3 = 0$ would have distance 1 from the origin.

(CMP3) is obvious.
(CMP4) follows immediately by the identification of $\mathcal{K}$ with

$$\{(a_0, a_1, a_2, a_3) \in \mathcal{P}_3\mathbb{R} \mid a_0^2 < a_1^2 + a_2^2 + a_3^2\}.$$

(CMP5) As in (CMP2) we introduce inhomogeneous coordinates. Let (a_0, a_1, a_2) be the inhomogeneous coordinates of K. The set of circles in $\mathcal{K}_p$ with homogeneous coordinates of the form $(b_0, b_1, b_2, 1)$ corresponds to some hyperplane H_p of $\mathbb{R}^3$. The hyperplanes H_q and H_r are defined analogously. The tangent spaces in question satisfy $\mathrm{T}_K\mathcal{K}_p \cap \mathrm{T}_K\mathcal{K}_q \cap \mathrm{T}_K\mathcal{K}_r = \{0\}$ if and only if the normal vectors of the hyperplanes H_p, H_q and H_r are linearly independent. These normal vectors have coordinates $n_p = (1, p_1, p_2)$, $n_q = (1, q_1, q_2)$ and $n_r = (1, r_1, r_2)$. Suppose, that n_p, n_q, n_r are linearly dependent. Then, without loss of generality, we may assume that $(1, p_1, p_2) = \lambda(1, q_1, q_2) + (1 - \lambda)(1, r_1, r_2)$ for some $\lambda \in \mathbb{R}$. In particular, the vectors $(p_1, p_2, 0)$, $(q_1, q_2, 0)$ and $(r_1, r_2, 0)$ lie on an affine line in $\mathbb{R}^3$. Thus, the points p, q and r are contained in an affine subplane of $\mathbb{R}^3$ with normal vector $(n_1, n_2, 0)$.

This contradicts the fact that p, q, r span an affine subplane with normal vector $(a_1, a_2, 1)$.

Remark. The relation $\mathrm{T}_K\mathcal{K}_p \cap \mathrm{T}_K\mathcal{K}_q \cap \mathrm{T}_K\mathcal{K}_r = \{0\}$ in (CMP5) implies that the submanifolds $\mathcal{K}_p$, $\mathcal{K}_q$, and $\mathcal{K}_r$ intersect pairwise transversally in K.

From now on we will deal with Moebius planes $\mathcal{M} = (S, \mathcal{K}, \mathcal{F})$ that satisfy the following axioms:

(SMP1) The point space S is a two-dimensional manifold and the space $\mathcal{K}$ of circles is a three-dimensional manifold.

(SMP2) The flag space $\mathcal{F}$ is a four-dimensional submanifold of $S \times \mathcal{K}$ such that the canonical projections $\pi_S : \mathcal{F} \to S : (p, K) \mapsto p$ and $\pi_{\mathcal{K}} : \mathcal{F} \to \mathcal{K} : (p, K) \mapsto K$ are submersions.

Note that by Proposition (3.1) the classical Moebius plane $\mathcal{M}_{\mathrm{cl}}$ does satisfy these axioms. The proofs of the next two results are quite similar to the proofs of Lemma (1.1) and of Theorem (1.2). For the proof of Theorem (3.3) it is essential that the projection $\pi_{\mathcal{K}}$ is a submersion.

LEMMA 3.2. *The sets $S_K = \pi_S(\pi_{\mathcal{K}}^{-1}(K))$ and $\mathcal{K}_p := \pi_{\mathcal{K}}(\pi_S^{-1}(p))$ are submanifolds of S and $\mathcal{K}$ having dimension* 1 *and* 2, *respectively.*

THEOREM 3.3. *Let $\mathcal{M} = (S, \mathcal{K}, \mathcal{F})$ be a Moebius plane that satisfies the axioms* (SMP1) *and* (SMP2). *If $K_1, K_2 \in \mathcal{K}$ with $S_{K_1} \cap S_{K_2} = \{p_1, p_2\}$ intersect transversally in S, then there are open neighborhoods U of (K_1, K_2) in $\mathcal{K} \times \mathcal{K}$ and U_i of p_i in S, $i = 1, 2$, such that for any pair $(L_1, L_2) \in U$ there is exactly one point $\beta_i(L_1, L_2) \in S_{K_1} \cap S_{K_2} \cap U_i$. Moreover, the maps $\beta_i : U \to U_i : (L_1, L_2) \mapsto \beta_i(L_1, L_2)$ are smooth.*

THEOREM 3.4. *Consider a Moebius plane $\mathcal{M} = (S, \mathcal{K}, \mathcal{F})$ that satisfies axioms (SMP1) and (SMP2). Let $p_1, p_2, p_3 \in S$ be three pairwise distinct points and let K be the unique circle that joins these three points. Assume that $\mathrm{T}_K\mathcal{K}_{p_1} \cap \mathrm{T}_K\mathcal{K}_{p_2} \cap \mathrm{T}_K\mathcal{K}_{p_3} = \{0\}$. Then there exist disjoint neighborhoods U_i of p_i, $i = 1, 2, 3$, such that the map $\alpha : U_1 \times U_2 \times U_3 \to \mathcal{K} : (q_1, q_2, q_3) \mapsto \alpha(q_1, q_2, q_3)$, which assigns to q_1, q_2, q_3 the unique circle through these points, is smooth.*

Proof. Since $\mathcal{F}$ is a submanifold of $S \times \mathcal{K}$, there exist pairwise disjoint open neighborhoods V_i of (p_i, K) in $S \times \mathcal{K}$ and submersions $\psi_i : V_i \to \mathbb{R}$ which vanish exactly on the sets $\mathcal{F} \cap V_i$, $i = 1, 2, 3$. Let

$$\psi : V_1 \times V_2 \times V_3 \to \mathbb{R}^3 : (x_1, x_2, x_3) \mapsto (\psi_1(x_1), \psi_2(x_2), \psi_3(x_3)),$$

$$\varphi : S^3 \times \mathcal{K} \to (S \times \mathcal{K})^3 : (q_1, q_2, q_3, L) \mapsto (q_1, L, q_2, L, q_3, L),$$

and $F = \psi \circ \varphi\big|_V$, where $V = \varphi^{-1}(V_1 \times V_2 \times V_3)$. Note that $F(q_1, q_2, q_3, L) = 0$ if and only if $L = \alpha(q_1, q_2, q_3)$. As in the proof of Theorem (1.2) it turns out that it is sufficient to show that for $v \in \mathrm{T}_K\mathcal{K}$ the condition

$$(0, v, 0, v, 0, v) \in \mathrm{T}_{(p_1,K)}\mathcal{F} \times \mathrm{T}_{(p_2,K)}\mathcal{F} \times \mathrm{T}_{(p_3,K)}\mathcal{F}$$

implies that $v = 0$. Since π_S is a submersion, we get $\{(0, u) \in \mathrm{T}_{(p_i,K)}\mathcal{F}\} = \{0\} \times \mathrm{T}_K\mathcal{K}_{p_i}$ for $i = 1, 2, 3$, and we infer that $v \in \mathrm{T}_K\mathcal{K}_{p_1} \cap \mathrm{T}_K\mathcal{K}_{p_2} \cap \mathrm{T}_K\mathcal{K}_{p_3} = \{0\}$. Again, the claim now follows by applying the implicit function theorem.

Remark. According to Proposition (3.1) the last two theorems are valid in particular for the classical Moebius plane $\mathcal{M}_{\mathrm{cl}}$.

References

1. Bödi, R.: Smooth stable planes, *Results. Math.* **31** (1997), 300–321.
2. Bröcker, T. and Jänich, K.: *Introduction to Differential Topology*, Cambridge University Press, 1987.
3. Buekenhout, F. (Ed.): *Handbook of Incidence Geometry*, North-Holland, Amsterdam, 1995.
4. Greenberg, M. J.: *Lectures on Algebraic Topology*, Benjamin, Menlo Park (Cal.), 1967.
5. Grundhöfer, T.: Ternary fields of compact projective planes, *Abh. Math. Sem. Univ. Hamburg* **57** (1987), 87–101.
6. Grundhöfer, T. and Hähl, H.: Fibrations of spheres by great spheres over division algebras and their differentiability, *J. Differential Geom.* **31** (1990), 357–363.
7. Hähl, H.: Differentiable fibrations of the $(2n-1)$-sphere by great $(n-1)$-spheres and their coordinatization over quasifields, *Results Math.* **12** (1987), 99–118.
8. Immervoll, S.: Glatte affine Ebenen und ihr projektiver Abschluß, Diploma thesis, Tübingen 1998.
9. Löwen, R.: Vierdimensionale stabile Ebenen, *Geom. Dedicata* **5** (1976), 239–294.
10. Löwen, R.: Topology and dimension of stable planes : On a conjecture of H. Freudenthal, *J. Reine Angew. Math.* **343** (1983), 108–122.
11. Löwen, R.: Compact spreads and compact translation planes over locally compact fields, *J. Geom.* **36** (1989), 110–116.
12. Massey, W. S.: *A Basic Course in Algebraic Topology*, Springer, New York, 1991.
13. Otte, J.: Differenzierbare Ebenen, Dissertation, Kiel, 1992.
14. Otte, J.: Smooth projective translation planes, *Geom. Dedicata* **58** (1995), 203-212.
15. Salzmann, H., Betten, D., Grundhöfer, T., Hähl, H., Löwen, R. and Stroppel, M.: *Compact Projective Planes,* De Gruyter, Berlin, 1995.
16. Strambach, K.: Der Kreisraum einer sphärischen Möbiusebene, *Monat. Math.* **78** (1974), 156–163.
17. Wölk, R. D.: Topologische Möbiusebenen, *Math. Z.* **93** (1966), 311–333.